CARTER-PAYNE HOMOMORPHISMS AND JANTZEN FILTRATIONS

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ABSTRACT. We prove a q-analogue of the Carter-Payne theorem in the case where the differences between the parts of the partitions are sufficiently large. We identify a layer of the Jantzen filtration which contains the image of these Carter-Payne homomorphisms and we show how these homomorphisms compose.

1. INTRODUCTION

The Iwahori-Hecke algebras of the symmetric groups are interesting algebras with a rich combinatorial representation theory. These algebras arise naturally in the representation theory of the general linear groups and they are important because they simultaneously extend and generalize the representation theory of the symmetric and general linear groups.

The representation theory of the Hecke algebra $\mathscr{H}_n$ closely parallels that of the symmetric groups. For each partition λ of n there is a Specht module S^λ. In the semisimple case the Specht modules give a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_n$-modules. When $\mathscr{H}_n$ is not semisimple it is an important problem to determine the structure of the Specht modules. The purpose of this paper is to construct explicit non-trivial homomorphisms between Specht modules in the non-semisimple case. Using this construction, we are then able to connect the image of the homomorphism and the Jantzen filtration of the corresponding Specht module.

The most striking result about homomorphisms between Specht modules of the symmetric groups is the *Carter-Payne Theorem* [**3**], which was proved by building on the famous paper of Carter and Lusztig [**2**]. A second proof of the Carter-Payne Theorem has recently been given by Fayers and Martin [**12**].

In this paper we are concerned with the Carter-Payne homomorphisms of the Iwahori-Hecke algebra of the symmetric group. To state our main results, let F be a field of characteristic $p \geq 0$ and fix a non-zero element $\zeta \in F$. Let $e > 1$ be minimal such that $1 + \zeta + \cdots + \zeta^{e-1} = 0$; set $e = 0$ if no such integer exists. Let $\mathscr{H}_n$ be the Hecke algebra of the symmetric group $\mathfrak{S}_n$, over F, with parameter ζ.

If $p > 0$ and $k > 0$ then define $\ell_p(k)$ to be the smallest positive integer such that $p^{\ell_p(k)} > k$. Now suppose that $\gamma > 0$ and λ and μ are partitions of n such that

$$\mu_i = \begin{cases} \lambda_i + \gamma, & i = a, \\ \lambda_i - \gamma, & i = z, \\ \lambda_i, & \text{otherwise}, \end{cases}$$

for some positive integers $a < z$. Let $h = \lambda_a - \lambda_z + z - a + \gamma$. Then λ and μ form an (e,p)**-Carter-Payne pair**, with parameters (a, z, γ), if $e > 1$ and either

a) $p = 0$, $\gamma < e$ and $h \equiv 0 \pmod{e}$, or,
b) $p > 0$ and $h \equiv 0 \pmod{ep^{\ell_p(\gamma^*)}}$, where $\gamma^* = \lfloor \frac{\gamma}{e} \rfloor$.

The Carter-Payne Theorem for an Iwahori-Hecke algebra of the symmetric group is the following result.

Theorem 1.1 (Carter and Payne [**3**] and Dixon [**6**])**.** *Suppose that F is a field of characteristic $p \geq 0$ and that λ and μ form an (e,p)-Carter-Payne pair. Then* $\operatorname{Hom}_{\mathscr{H}_n}(S^\lambda, S^\mu) \neq 0$.

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For the symmetric groups (that is, when $q = 1$) this theorem is a classical result of Carter and Payne [**3**]. The full q-analogue of this result for the Iwahori-Hecke algebra $\mathscr{H}_n$ was recently established in the unpublished thesis of Dixon [**6**]. Dixon's proof follows the original arguments of Carter and Lusztig [**2**] and Carter and Payne [**3**]. He works with the quantum hyperalgebra U of the general linear group and he proves that if λ and μ form an (e,p)-Carter-Payne then $\operatorname{Hom}_U(\Delta^\lambda,\Delta^\mu)$ is one dimensional, where Δ^ν is the Weyl module of U indexed by the partition ν. As $\dim\operatorname{Hom}_{\mathscr{H}_n}(S^\lambda,S^\mu) \geq \dim\operatorname{Hom}_U(\Delta^\lambda,\Delta^\mu)$, with equality if $q \neq -1$, this implies Theorem 1.1 for arbitrary q.

The Carter-Payne homomorphisms are very useful and important maps. Unfortunately little is known about them in general except that they exist. In this paper we concentrate on separated Carter-Payne pairs, where an (e,p)-Carter-Payne pair (λ,μ) with parameters (a,z,γ) is **separated** if $\lambda_r - \lambda_{r+1} \geq \gamma$ for $a < r \leq z$. We begin by giving two new and very explicit descriptions of Carter-Payne homomorphisms $\theta_{\lambda\mu}\colon S^\lambda \longrightarrow S^\mu$ when λ and μ form a separated Carter-Payne pair. We then use the new descriptions to prove the following two results, which were known previously only for the symmetric group algebra when $\gamma = 1$ [**11**].

Theorem 1.2. *Suppose that λ, μ and σ are partitions of n such that λ and σ form a separated (e,p)-Carter-Payne pair with parameters (a,y,γ) and that σ and μ form a separated (e,p)-Carter-Payne pair with parameters (y,z,γ), where $a < y < z$ and $\gamma > 0$. Then λ and μ form a separated (e,p)-Carter-Payne pair with parameters (a,z,γ) and $\theta_{\lambda\sigma}\theta_{\sigma\mu} = \theta_{\lambda\mu}$.*

To state our next result let $S^\mu = J^0(S^\mu) \supseteq J^1(S^\mu) \supseteq J^2(S^\mu) \supseteq \dots$ be the Jantzen filtration of S^μ (see Section 2.6), and for $0 \neq h \in \mathbb{Z}$ define

$$\mathit{val}_{e,p}(h) = \begin{cases} p^{\mathit{val}_p(h)}, & \text{if } e \mid h,\\ 0, & \text{otherwise,}\end{cases}$$

where val_p is the usual p-adic valuation map (and we set $\mathit{val}_0(h) = 0$ when $p = 0$). Our second main result is the following.

Theorem 1.3. *Suppose that $p \geq 0$ and that λ and μ form a separated (e,p)-Carter-Payne pair with parameters (a,z,γ). Then*

$$\operatorname{Im}\theta_{\lambda\mu} \subseteq J^\delta(S^\mu),$$

where $\delta = \mathit{val}_{e,p}(\lambda_a - \lambda_z + z - a + \gamma) - \mathit{val}_{e,p}(\gamma)$.

The key observation in our construction of the Carter-Payne homomorphisms, which is due to Ellers and Murray [**10**], is that the Specht modules S^λ and S^μ both appear in the restriction of a Specht module S^ν of $\mathscr{H}_{n+\gamma}$. Starting from this observation we are able show that the Carter-Payne homomorphism $\theta_{\lambda\mu}\colon S^\lambda \longrightarrow S^\mu$ is induced by an $\mathscr{H}_n$-module endomorphism of S^ν which is given by right multiplication by a polynomial in the Jucys-Murphy elements $L_{n+1},\dots,L_{n+\gamma}$ of $\mathscr{H}_{n+\gamma}$. Using this description of the Carter-Payne maps we are able to prove the two theorems above as well as describe these maps as explicit linear combinations of semistandard homomorphisms. Thus we give a new proof of Theorem 1.1, when λ and μ are a separated pair, which takes place entirely within the Hecke algebra.

We now describe the contents of this paper in more detail. Section 2 sets up the basic notation and machinery that is used throughout the paper. In Theorem 2.7 and Theorem 2.8 we show that if (λ,μ) is a separated (e,p)-Carter-Payne pair then the corresponding Carter-Payne homomorphism is given by right multiplication by a polynomial in the Jucys-Murphy elements of $\mathscr{H}_{n+\gamma}$. We prove these results by writing the Carter-Payne homomorphism $\theta_{\lambda\mu}$ as an explicit linear combination of semistandard homomorphisms. These results are proved modulo a result which describes how the Jucys-Murphy elements

act on the Specht modules (Proposition 2.5) and a technical result which allows us to divide our maps by certain polynomial coefficients when $p > 0$ (Lemma 3.24). Using these results we prove our two main theorems about composing Carter-Payne homomorphisms and the connection between these maps and the Jantzen filtration. Section 3 is the computational heart of the paper which proves the detailed technical results which describe the action of the Jucys-Murphy elements on the Specht modules which are need to prove our main theorems. The results in this section are likely to be of independent interest.

2. CARTER-PAYNE HOMOMORPHISMS AND JUCYS-MURPHY ELEMENTS

In this section we define the Hecke algebra and the Specht modules and reduce the proofs of our main results to some technical statements which are proved in the next section.

2.1. **The Hecke algebra.** For each integer $n > 0$ let $\mathfrak{S}_n$ be the symmetric group of degree n. The symmetric group $\mathfrak{S}_n$ is generated by the simple transpositions $s_1, s_2, \dots, s_{n-1}$, where $s_i = (i, i+1)$ for $1 \le i < n$. If $w \in \mathfrak{S}_n$ then $s_{i_1} \dots s_{i_k}$ is a **reduced expression** for w if $w = s_{i_1} \dots s_{i_k}$ and k is minimal with this property. In this case, the **length** of w is $\ell(w) = k$.

Suppose that q is an indeterminate over $\mathbb{Z}$ and let $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials in q. The **generic Iwahori-Hecke algebra** of $\mathfrak{S}_n$ is the unital associative $\mathcal{Z}$-algebra $\mathscr{H}_n^{\mathcal{Z}}$ with generators $T_1, \dots, T_{n-1}$ which are subject to the relations

$$(T_i - q)(T_i + 1) = 0, \quad T_j T_{j+1} T_j = T_{j+1} T_j T_{j+1} \quad \text{and} \quad T_i T_j = T_j T_i,$$

where $1 \le i < n$, $1 \le j < n-1$ and $|i - j| \ge 2$. The Hecke algebra $\mathscr{H}_n$ is free as an $\mathcal{Z}$-module with basis $\{ T_w \mid w \in \mathfrak{S}_n \}$, where $T_w = T_{i_1} \dots T_{i_k}$ and $s_{i_1} \dots s_{i_k}$ is a reduced expression for w; see, for example, [**17**, Chapt. 1].

Now suppose that R is an arbitrary ring and that q_R is an invertible element of R. Define $\mathscr{H}_n^R(q_R) = \mathscr{H}_n^{\mathcal{Z}} \otimes_{\mathcal{Z}} R$, where we consider R as a $\mathcal{Z}$-algebra by letting q act as multiplication by q_R. We say that $\mathscr{H}_n$ is obtained from $\mathscr{H}_n^{\mathcal{Z}}$ by **specialization** at $q = q_R$. By the remarks above, $\mathscr{H}_n^R(q_R)$ is a unital associative R-algebra which is free as an R-module with basis $\{ T_w \otimes 1 \mid w \in \mathfrak{S}_n \}$. Typically, we abuse notation and write T_w instead of $T_w \otimes 1$, for $w \in \mathfrak{S}_n$.

In this paper we are most interested in the algebra $\mathscr{H}_n = \mathscr{H}_n^F(\zeta)$, where F is a field of characteristic $p \ge 0$ and $0 \ne \zeta \in F$. Define

$$e = \min \{ f \ge 2 \mid 1 + \zeta + \dots + \zeta^{f-1} = 0 \},$$

and set $e = 0$ if $1 + \zeta + \dots + \zeta^{f-1} \ne 0$ for all $f \ge 2$. Then $\mathscr{H}_n$ is (split) semisimple if and only if $e > n$ or $e = 0$; see, for example, [**17**, Cor. 3.24]). Henceforth, we assume that $2 \le e \le n$. In particular, $\mathscr{H}_n$ is not semisimple.

Observe that if $\zeta = 1$ then $e = p$ and $\mathscr{H}_n \cong F\mathfrak{S}_n$. If $\zeta \ne 1$ then ζ is a primitive e^{th} root of unity in F.

2.2. **Tableaux combinatorics.** A **composition** of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers which sum to n and λ is a **partition** if $\lambda_1 \ge \lambda_2 \ge \dots$. The diagram of a partition λ is the set $\mathbb{D}(\lambda) = \{ (r, c) \mid 1 \le c \le \lambda_r, \text{for } r \ge 1 \}$. A (row standard) λ-**tableau** is a map $\mathsf{S} : \mathbb{D}(\lambda) \longrightarrow \mathbb{N}$ such that $\mathsf{S}(r, c) \le \mathsf{S}(r, c')$, whenever $c \le c'$. We identify a λ-tableau with a labeling of the diagram of λ by $\mathbb{N}$, and in this way we can talk of the rows and columns of S. A λ-tableau S is:

a) **semistandard** if the entries in S are strictly increasing down columns.
b) **standard** if $\mathsf{S} : \mathbb{D}(\lambda) \longrightarrow \{1, 2, \dots, n\}$ is a bijection and the entries in S are strictly increasing down columns;

A λ-tableau has **type** $\mu = (\mu_1, \mu_2, \dots)$ if it has μ_i entries equal to i, for $i \ge 1$. If S is a λ-tableau let $\operatorname{Shape}(\mathsf{S}) = \lambda$ and if $k \ge 0$ let $\mathsf{S}_{\downarrow k}$ be the subtableau of S containing the numbers $1, 2, \dots, k$.

The following notation will help us keep track of certain entries in our tableaux.

Notation. Suppose that S is a tableau and that X and R are sets of positive integers. Let S_R^X be the number of entries in row r of S which are equal to some x, for some $r \in R$ and some $x \in X$. We further abbreviate this notation by setting $\mathsf{S}_{>r}^{\le x} = \mathsf{S}_{(r,\infty)}^{[1,x]}$, $\mathsf{S}_r^x = \mathsf{S}_{\{r\}}^{\{x\}}$ and so on.

Let $\mathcal{T}(\lambda,\mu)$ be the set of λ-tableau of type μ and $\mathcal{T}_0(\lambda,\mu)$ the set of semistandard λ-tableaux of type μ. Let $\operatorname{Std}(\lambda) = \mathcal{T}_0(\lambda,(1^n))$ be the set of standard tableaux and $\operatorname{RStd}(\lambda) = \mathcal{T}(\lambda,(1^n))$ the set of tableaux of type (1^n). The **initial λ-tableau** is the standard λ-tableau $\mathfrak{t}^\lambda$ obtained by entering the numbers $1, 2, \dots, n$ in increasing order, from left to right, along the rows of $\mathbb{D}(\lambda)$.

If $\mathfrak{s}$ is a tableau and $\mathfrak{s}(r,c) = k$ then define $\operatorname{row}_\mathfrak{s}(k) = r$. For any subset $I \subseteq \{1, 2, \dots, n\}$, the entries in I are in **row order** in $\mathfrak{s}$ if $\operatorname{row}_\mathfrak{s}(i) \le \operatorname{row}_\mathfrak{s}(j)$ whenever $i < j \in I$. For example, $\mathfrak{t}^\lambda$ is the unique λ-tableau which has $1, 2, \dots, n$ in row order.

There is an action of $\mathfrak{S}_n$ on $\operatorname{RStd}(\lambda)$, from the right, given by defining $\mathfrak{s}w$ to be the λ-tableau obtained from $\mathfrak{s}$ by acting on the entries of $\mathfrak{s}$ by w and then reordering the entries in each row, for $\mathfrak{s} \in \operatorname{RStd}(\lambda)$ and $w \in \mathfrak{S}_n$. If $\mathfrak{s} \in \operatorname{RStd}(\lambda)$ define $d(\mathfrak{s})$ to be the unique element of $\mathfrak{S}_n$ of minimal length such that $\mathfrak{s} = \mathfrak{t}^\lambda d(\mathfrak{s})$; such an element exists, for example, by [**17**, Prop. 3.3]. The permutation $d(\mathfrak{s})$ is the unique element of $\mathfrak{S}_n$ such that $\mathfrak{s} = \mathfrak{t}^\lambda d(\mathfrak{s})$ and $(i)d(\mathfrak{s}) < (j)d(\mathfrak{s})$ whenever $i < j$ lie in the same row of $\mathfrak{s}$. Let $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \dots$ be the **Young subgroup** of $\mathfrak{S}_n$ associated to λ.

2.3. **Specht modules.** For each pair of tableaux $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda)$, for λ a partition of n, let $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})^{-1}} m_\lambda T_{d(\mathfrak{t})}$, where

$$m_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w.$$

Murphy showed that $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda), \text{ for } \lambda \text{ a partition of } n \,\}$ is a basis of $\mathscr{H}_n$ [**17**, **18**]. The basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ is a cellular basis of $\mathscr{H}_n$ with respect to the dominance ordering where if λ and μ are partitions then $\mu \trianglerighteq \lambda$ if

$$\sum_{i=1}^{j} \mu_i \ge \sum_{i=1}^{j} \lambda_i,$$

for all $j \ge 1$. Write $\mu \triangleright \lambda$ if $\mu \trianglerighteq \lambda$ and $\mu \ne \lambda$. Let $\mathscr{H}^{\triangleright\lambda}$ be the two-sided ideal of $\mathscr{H}_n$ with basis $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\mu) \text{ for some } \mu \triangleright \lambda \,\}$.

Fix a partition λ of n. The **Specht module** S_F^λ is the $\mathscr{H}_n$-submodule of $\mathscr{H}_n/\mathscr{H}^{\triangleright\lambda}$ generated by $m_\lambda + \mathscr{H}^{\triangleright\lambda}$. For every tableau $\mathfrak{s} \in \operatorname{RStd}(\lambda)$ define $m_\mathfrak{s} = m_\lambda T_{d(\mathfrak{s})} + \mathscr{H}^{\triangleright\lambda}$. Then $m_\mathfrak{s} \in S^\lambda$ and $\{\, m_\mathfrak{t} \mid \mathfrak{t} \in \operatorname{Std}(\lambda) \,\}$ is a basis of S_F^λ by, for example, [**17**, Prop. 3.22]. This construction of the Specht module works over an arbitrary ring. In particular, we have a Specht module $S_\mathcal{Z}^\lambda$ for the generic Hecke algebra $\mathscr{H}_n^\mathcal{Z}$ and $S_F^\lambda \cong S_\mathcal{Z}^\lambda \otimes_\mathcal{Z} F$ as $\mathscr{H}_n$-modules. Usually, we write $S^\lambda = S_F^\lambda$ unless we want to emphasize the ring that S^λ is defined over.

We emphasize, for the readers convenience, that throughout this paper we follow [**17**] and work with the Specht modules that arise as the cell modules for the Murphy basis. These modules are *dual* to the classically defined Specht modules considered in [**7**, **13**]. Our results can be translated into the corresponding results for the classical Specht modules by conjugating the partitions involved and taking duals; see, for example, [**16**, Lemma 3.4].

Define the **Jucys-Murphy elements** $L_1, \dots, L_n$ of $\mathscr{H}_n$ by setting $L_1 = q^{-1}T_1$ and $L_{k+1} = q^{-1}T_k(1 + L_kT_k)$. Then $L_1, \dots, L_n$ generate a commutative subalgebra of $\mathscr{H}_n$;

see, for example, [**17**, Prop. 3.26]. The Jucys-Murphy elements L_k are important for us because they act on the Specht modules via triangular matrices.

If R is any ring, $a \in R$ and $k \in \mathbb{Z}$ then the **Gaussian integer** $[k]_a$ is

$$[k]_a = \lim_{t\to a} \tfrac{t^k-1}{t-1},$$

where t is an indeterminate over R. If $k \geq 0$ set $[0]^!_a = 1$ and let $[k]^!_a = [k-1]^!_a[k]_a$. We are most interested in these scalars when $R = \mathcal{Z}$ and $r = q$, so we set $[k] = [k]_q$ and $[k]^! = [k]^!_q$, for $k \in \mathbb{Z}$.

2.4. **Constructing Carter-Payne homomorphisms.** Suppose that λ is a partition of n and let $M^\lambda = m_\lambda \mathscr{H}_n$ be the corresponding permutation module for $\mathscr{H}_n$. Then M^λ has basis $\{\, m_\lambda T_{d(\mathfrak{t})} \mid \mathfrak{t} \in \operatorname{RStd}(\lambda)\,\}$ and there is a surjective homomorphism $\pi_\lambda : M^\lambda \longrightarrow S^\lambda$ given by $\pi_\lambda(m_\lambda T_{d(\mathfrak{t})}) = m_\mathfrak{t}$, for $\mathfrak{t} \in \operatorname{RStd}(\lambda)$.

Now if μ is a partition of n and $\mathfrak{t} \in \operatorname{Std}(\mu)$, define $\lambda(\mathfrak{t})$ to be the μ-tableau obtained by replacing each entry in $\mathfrak{t}$ by its row index in $\mathfrak{t}^\lambda$. If T is a μ-tableau of type λ define

$$m_\mathsf{T} = \sum_{\substack{\mathfrak{t}\in\operatorname{RStd}(\mu)\\ \lambda(\mathfrak{t})=\mathsf{T}}} m_\mathfrak{t}.$$

By definition $m_\mathsf{T} \in S^\mu$.

If $\mathsf{T} \in \mathcal{T}_0(\mu,\lambda)$ let $\varphi_\mathsf{T} \in \operatorname{Hom}_{\mathscr{H}_n}(M^\lambda, S^\mu)$ be the homomorphism determined by $\varphi_\mathsf{T}(m_\lambda) = m_\mathsf{T}$ and let $\mathcal{H}om_{\mathscr{H}_n}(M^\lambda, S^\mu)$ be the subspace of $\operatorname{Hom}_{\mathscr{H}_n}(M^\lambda, S^\mu)$ spanned by $\{\, \varphi_\mathsf{T} \mid T \in \mathcal{T}_0(\mu,\lambda)\,\}$. Let $\mathcal{H}om_{\mathscr{H}_n}(S^\lambda, S^\mu)$ be the space of homomorphisms $\varphi \in \operatorname{Hom}_{\mathscr{H}_n}(S^\lambda, S^\mu)$ such that $\pi_\lambda\varphi \in \mathcal{H}om_{\mathscr{H}_n}(M^\lambda, S^\mu)$.

To reprove Theorem 1.1 for the separated (e,p)-Carter-Payne pair (λ,μ) we use the following result. This is purely a matter of notational convenience as the proof that we give can be made to work without making use of this proposition (cf. the proof of Theorem 1.2 in Section 2.5 below).

Proposition 2.1. *Suppose that λ and μ are partitions of m such that $\lambda_i = \mu_i$, whenever $1 \leq i < a$ or $i > z$, for some integers $a < z$. Define $\hat\lambda = (\lambda_a, \lambda_{a+1}, \dots, \lambda_z)$ and $\hat\mu = (\mu_a, \mu_{a+1}, \dots, \mu_z)$ and let $n = \hat\lambda_a + \cdots + \hat\lambda_z = \hat\mu_a + \cdots + \hat\mu_z$. Then*

$$\mathcal{H}om_{\mathscr{H}_m}(S^\lambda, S^\mu) \cong_F \mathcal{H}om_{\mathscr{H}_n}(\mathsf{S}^{\hat\lambda}, \mathsf{S}^{\hat\mu}).$$

Proof. This follows from (the proof of) [**16**, Theorem 3.2 and Lemma 3.4]; cf. [**9**, Prop. 10.4]. □

Therefore, when constructing Carter-Payne homomorphisms it is enough to show that $\mathcal{H}om_{\mathscr{H}_n}(\mathsf{S}^\lambda, \mathsf{S}^\mu) \neq 0$ for partitions λ and μ of n which form a separated (e,p)-Carter-Payne pair with parameters $a = 1$, $z = \max\{\, i > 0 \mid \lambda_i \neq 0\,\}$ and $\gamma > 0$. For the rest of Section 2.4 we fix such a pair. We define ν to be the partition of $n+\gamma$ given by

$$\nu_i = \begin{cases} \lambda_i + \gamma, & \text{if } i = 1,\\ \lambda_i, & \text{otherwise.}\end{cases}$$

There is a natural embedding $\mathscr{H}_n \hookrightarrow \mathscr{H}_{n+\gamma}$. Thus we can consider any $\mathscr{H}_{n+\gamma}$-module as an $\mathscr{H}_n$-module by restriction. We need the following well-known result – it is an easy corollary of [**17**, Prop. 6.1].

Lemma 2.2. *As an $\mathscr{H}_n$-module the Specht module S^ν has a filtration*

$$S^\nu = M_0 \supset M_1 \supset \cdots \supset M_k \supset 0,$$

such that $M_i/M_{i+1} \cong S^{\tau_i}$, for some partition τ_i of n, for $0 \leq i \leq k$. Moreover $S^\lambda \cong M_0/M_1$, $S^\mu \cong M_k$, M_1 has basis $\{\, m_\mathfrak{t} \mid \mathfrak{t} \in \operatorname{Std}(\nu)$ and $\operatorname{Shape}(\mathfrak{t}_{\downarrow n}) \neq \lambda\,\}$, and M_k has basis $\{\, m_\mathfrak{t} \mid \mathfrak{t} \in \operatorname{Std}(\nu)$ and $\mathfrak{t}_{\downarrow n} \in \operatorname{Std}(\mu)\,\}$.

Hence, following Ellers and Murray [**10**, § 3], we have the following elementary but very useful observation.

Corollary 2.3. *Let ν be the partition of $n+\gamma$ defined above and suppose that there exists a non-zero homomorphism $\theta \in \operatorname{End}_{\mathscr{H}_n}(S^\nu)$ such that $M_1 \subseteq \ker(\theta)$ and $\operatorname{Im}(\theta) \subseteq M_k$. Then $\operatorname{Hom}_{\mathscr{H}_n}(S^\lambda, S^\mu) \neq 0$.*

Set $c_i = \nu_i - i$, for $1 \le i \le z$. (Thus, c_i is the content of the i^{th} removable node of ν; see the remarks before Lemma 3.1.) Now define

$$L_{\lambda\mu} = \prod_{i=1}^{z-1}\prod_{j=1}^{\gamma}\Big(L_{n+j} - [c_i]\Big).$$

Lemma 2.4. *Suppose that $1 \le k < n+\gamma$ and $k \neq n$. Then $T_k L_{\lambda\mu} = L_{\lambda\mu} T_k$.*

Proof. By Lemma 3.3, below, if $k \neq n$ then T_k commutes with $(L_{n+1}-c)\dots(L_{n+\gamma}-c)$, for any $c \in F$. □

Hence, right multiplication by $L_{\lambda\mu}$ induces an $\mathscr{H}_n$-endomorphism of S^ν. The following definitions allow us to describe this map and (if necessary) to modify it so as to produce an endomorphism $\theta_{\lambda\mu}$ which satisfies the conditions of Lemma 2.3.

Let η be a partition of n. Write $\eta \subseteq \nu$ if $\mathbb{D}(\eta) \subseteq \mathbb{D}(\nu)$; equivalently, $\eta_i \le \nu_i$, for $i \ge 1$. If $\eta \subseteq \nu$, define $\eta + 1^\gamma = (\eta_1, \dots, \eta_k, 0^{z-k}, 1^\gamma)$, where $k = \max\{\, i \mid \eta_i > 0\,\}$. Define $\mathfrak{t}^\nu_\eta$ to be the standard tableau which agrees with $\mathfrak{t}^\eta$ where $\mathbb{D}(\eta)$ and $\mathbb{D}(\nu)$ coincide, with the numbers $n+1, \dots, n+\gamma$ entered in row order in the remaining nodes of $\mathbb{D}(\nu)$. A ν-tableau $\mathfrak{t}$ is **almost initial** if $\mathfrak{t} = \mathfrak{t}^\nu_\eta$, for some partition η of n.

Now suppose that η is a partition of n such that $\eta \subseteq \nu$. Define

$$\mathcal{T}_0^\nu(\mu,\eta) = \{\, \mathsf{S} \in \mathcal{T}_0(\nu, \eta+1^\gamma) \mid \operatorname{Shape}(\mathsf{S}_{\downarrow z}) = \mu \,\}.$$

That is, $\mathcal{T}_0^\nu(\mu,\eta)$ is the set of semistandard ν-tableaux of type $\eta + 1^\gamma$ obtained by adding nodes labeled $z+1, \dots, z+\gamma$ to the bottom of a semistandard μ-tableaux of type η. Similarly, if $\eta \subseteq \nu$ let $\operatorname{Std}_\eta(\nu) = \{\, \mathfrak{t} \in \operatorname{Std}(\nu) \mid \operatorname{Shape}(\mathfrak{t}_{\downarrow n}) = \eta \,\}$.

The following elegant result will allow us to construct Carter-Payne homomorphisms. It will be proved with less elegance in the following sections. The number $\mathsf{S}_r^{(r,z]}$, which is the number of entries in row r of S contained in $(r,z]$, is defined in Section 2.2.

Proposition 2.5. *Suppose that $\eta \subset \nu$ is a partition of n. Then there exists an integer C such that in $S^\nu_{\mathcal{Z}}$*

$$m_{\mathfrak{t}^\nu_\eta} L_{\lambda\mu} = q^C \sum_{\mathsf{S} \in \mathcal{T}_0^\nu(\mu,\eta)} \prod_{r=1}^{z-1}\Big([\mathsf{S}_r^{(r,z]}]^! \prod_{j=0}^{\gamma - \mathsf{S}_r^{(r,z]} - 1} [c_z - c_r - j]\Big) m_{\mathsf{S}}.$$

Proof. This is the special case of Proposition 3.19 below, obtained by setting $k = \gamma$ and $y = 1$. □

2.6. **Example** Suppose that $\lambda = (4,4,2)$ and $\mu = (6,4)$. Then λ and μ form a $(6,0)$-Carter-Payne pair with parameters $(a,z,\gamma) = (1,3,2)$. Applying the definitions,

$$L_{\lambda\mu} = (L_{12} - [5])(L_{12} - [2])(L_{11} - [5])(L_{11} - [2]).$$

Identifying the tableau S with m_{S}, direct computation (or Proposition 2.5) shows that

$$
\begin{aligned}
\begin{array}{cccccc}1&2&3&4&11&12\\5&6&7&8\\9&10\end{array}\ L_{\lambda\mu} &= \begin{array}{cccccc}1&1&1&1&4&5\\2&2&2&2\\3&3\end{array}\ L_{\lambda\mu} = q^{2}[2][2]\ \begin{array}{cccccc}1&1&1&1&2&2\\2&2&3&3\\4&5\end{array} - q^{-1}[2][3]\ \begin{array}{cccccc}1&1&1&1&2&3\\2&2&2&3\\4&5\end{array}\\
&\qquad + q^{-5}[2][3][4]\ \begin{array}{cccccc}1&1&1&1&3&3\\2&2&2&2\\4&5\end{array},\\
\begin{array}{cccccc}1&2&3&4&5&11\\6&7&8&12\\9&10\end{array}\ L_{\lambda\mu} &= \begin{array}{cccccc}1&1&1&1&1&4\\2&2&2&5\\3&3\end{array}\ L_{\lambda\mu} = -q^{-2}[2][6]\ \begin{array}{cccccc}1&1&1&1&1&2\\2&2&3&3\\4&5\end{array} + q^{-5}[3][6]\ \begin{array}{cccccc}1&1&1&1&1&3\\2&2&2&3\\4&5\end{array},\\
\begin{array}{cccccc}1&2&3&4&5&11\\6&7&8&9\\10&12\end{array}\ L_{\lambda\mu} &= \begin{array}{cccccc}1&1&1&1&1&4\\2&2&2&2\\3&5\end{array}\ L_{\lambda\mu} = q^{-2}[3][6]\ \begin{array}{cccccc}1&1&1&1&1&2\\2&2&2&3\\4&5\end{array} - q^{-6}[3][4][6]\ \begin{array}{ccccccc}1&1&1&1&1&1&3\\2&2&2&2\\4&5\end{array},\\
\begin{array}{cccccc}1&2&3&4&5&6\\7&8&11&12\\9&10\end{array}\ L_{\lambda\mu} &= \begin{array}{cccccc}1&1&1&1&1&1\\2&2&4&5\\3&3\end{array}\ L_{\lambda\mu} = q^{-5}[2][6][7]\ \begin{array}{cccccc}1&1&1&1&1&1\\2&2&3&3\\4&5\end{array},\\
\begin{array}{cccccc}1&2&3&4&5&6\\7&8&9&11\\10&12\end{array}\ L_{\lambda\mu} &= \begin{array}{cccccc}1&1&1&1&1&1\\2&2&2&4\\3&5\end{array}\ L_{\lambda\mu} = -q^{-6}[3][6][7]\ \begin{array}{cccccc}1&1&1&1&1&1\\2&2&2&3\\4&5\end{array},\\
\begin{array}{cccccc}1&2&3&4&5&6\\7&8&9&10\\11&12\end{array}\ L_{\lambda\mu} &= \begin{array}{cccccc}1&1&1&1&1&1\\2&2&2&2\\4&5\end{array}\ L_{\lambda\mu} = q^{-6}[3][4][6][7]\ \begin{array}{cccccc}1&1&1&1&1&1\\2&2&2&2\\4&5\end{array}.
\end{aligned}
$$

Using these calculations we invite the reader to check that right multiplication by $L_{\lambda\mu}$ induces an $\mathscr{H}_{10}^{\mathbb{C}}$-module homomorphism $S^{\lambda} \to S^{\mu}$ when $\zeta = \exp(2\pi i/6) \in \mathbb{C}$ (so that $e = 6$). $\Diamond$

Using Proposition 2.5, we can now give a second proof of Theorem 1.1 from the introduction for our pair (λ, μ). We treat the cases $p = 0$ and $p > 0$ separately because the proof when $p > 0$ contains an additional subtlety.

Theorem 2.7. *Suppose that $p = 0$ and that λ and μ form a separated $(e, 0)$-Carter-Payne pair with parameters (a, z, γ). Then*

$$\mathscr{H}om_{\mathscr{H}_n}(S^{\lambda}, S^{\mu}) \neq 0.$$

Proof. By Proposition 2.1 it is enough to consider the case when $a = 1$ and $\lambda_r = 0$ when $r > z$. Since λ and μ form a Carter-Payne pair we have, by assumption, that $\gamma < e$ and

$$\lambda_1 - \lambda_z + z - 1 + \gamma = c_1 - c_z \equiv 0 \pmod{e}.$$

In particular, $[c_z - c_1]_\zeta = 0$ in F.

Suppose that $\mathfrak{t} \in \operatorname{Std}(\nu)$ and let $\eta = \operatorname{Shape}(\mathfrak{t}_{\downarrow n})$. Then in S^{ν} we have $m_{\mathfrak{t}} = m_{\mathfrak{t}^{\nu}_{\eta}} T_w$, for some $w \in \mathfrak{S}_n \times \mathfrak{S}_\gamma$. Therefore, by specializing $q = \zeta$ in Proposition 2.5 and using Lemma 2.4, we have

$$m_{\mathfrak{t}} L_{\lambda\mu} = \zeta^{C} \sum_{\mathsf{S} \in \mathcal{T}_0^{\nu}(\mu,\eta)} \prod_{r=1}^{z-1} \Big([\mathsf{S}_r^{(r,z]}]_\zeta^! \prod_{j=0}^{\gamma - \mathsf{S}_r^{(r,z]} - 1} [c_z - c_r - j]_\zeta \Big) m_{\mathsf{S}} T_w,$$

for some $C \in \mathbb{Z}$. Recall that as an $\mathscr{H}_n$-module S^{ν} has a Specht filtration $S^{\nu} = M_0 \supset M_1 \supset \cdots \supset M_k \supset 0$ with $S^{\lambda} \cong M_0/M_1$ and $S^{\mu} \cong M_k$ by Lemma 2.2. Moreover, M_k is spanned by the $m_{\mathfrak{s}}$, for $\mathfrak{s} \in \operatorname{Std}(\nu)$ with $\operatorname{Shape}(\mathfrak{s}_{\downarrow n}) = \mu$. Therefore, the last displayed equation shows that $m_{\mathfrak{t}} L_{\lambda\mu} \in M_k$ for $\mathfrak{t} \in \operatorname{Std}(\nu)$.

Next suppose that $\mathfrak{t} \in \operatorname{Std}_\eta(\nu)$ and $m_{\mathfrak{t}} \in M_1$. Then $\eta \neq \lambda$ by Lemma 2.2. Consequently, if $\mathsf{S} \in \mathcal{T}_0^{\nu}(\mu, \eta)$ then $\mathsf{S}_1^{(1,z]} < \gamma$ and $[c_z - c_1]_\zeta$ divides the coefficient of m_{S} in $m_{\mathfrak{t}} L_{\lambda\mu}$. That is, $m_{\mathfrak{t}} L_{\lambda\mu} = 0$ since $[c_z - c_1]_\zeta = 0$ in F.

By the last two paragraphs, and Lemma 2.3, right multiplication by $L_{\lambda\mu}$ induces an $\mathscr{H}_n$-module homomorphism from S^{λ} to S^{μ}. Suppose that $\mathfrak{t} = \mathfrak{t}^{\nu}_{\lambda}$. Then there exists a semistandard tableau $\mathsf{S} \in \mathcal{T}_0^{\nu}(\mu, \lambda)$ with $\mathsf{S}_r^{(r,z]} = \gamma$, for $1 \leq r < z$. This is the unique semistandard tableau $\mathsf{S} \in \mathcal{T}_0^{\nu}(\mu, \lambda)$ such that row r contains γ entries equal to $r + 1$, for $1 \leq r < z$. The coefficient of m_{S} in $m_{\mathfrak{t}^{\nu}_{\lambda}} L_{\lambda\mu}$ is $\zeta^{C}\big([\gamma]_\zeta\big)^{z-1} \neq 0$, so that $m_{\mathfrak{t}^{\nu}_{\lambda}} L_{\lambda\mu} \neq 0$ as required.

We have now shown that right multiplication on S^ν by $L_{\lambda\mu}$ induces a non-zero map $\theta_{\lambda\mu}\colon S^\lambda \longrightarrow S^\mu$. It remains to show that $\theta_{\lambda\mu} \in \mathscr{H}om_{\mathscr{H}_n}(S^\lambda, S^\mu)$. However, from what we have proved it follows that

$$\pi_\lambda\theta_{\lambda\mu} = \zeta^C \sum_{\mathsf{S}\in\mathcal{T}_0(\mu,\lambda)} \prod_{r=1}^{z-1} \Big([\mathsf{S}_r^{(r,z)}]_\zeta^! \prod_{j=0}^{\gamma-\mathsf{S}_r^{(r,z]}-1} [c_z - c_r - j]_\zeta \Big) \varphi_{\mathsf{S}}.$$

Therefore, $\theta_{\lambda\mu} \in \mathscr{H}om_{\mathscr{H}_n}(S^\lambda, S^\mu)$ as claimed. □

We now consider the case when $p > 0$. The argument is essentially the same as in the case when $p = 0$. There is a technical difficulty, however, because in general multiplication by $L_{\lambda\mu}$ induces the zero homomorphism from S^λ to S^μ.

Theorem 2.8. *Suppose that $p > 0$ and that λ and μ form a separated (e, p)-Carter-Payne pair with parameters (a, z, γ). Then*

$$\mathscr{H}om_{\mathscr{H}_n}(S^\lambda, S^\mu) \neq 0$$

Proof. As in Theorem 2.7, we may assume that $a = 1$ and $\lambda_r = 0$ for $r > z$. We first consider the Specht module $S^\nu_{\mathcal{Z}}$ for the generic Hecke algebra $\mathscr{H}^{\mathcal{Z}}_{n+\gamma}$ defined over $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$. Suppose that $\mathfrak{t} \in \mathrm{Std}(\nu)$ and set $\eta = \mathrm{Shape}(\mathfrak{t}_{\downarrow n})$ so that $m_\mathfrak{t} = m_{\mathfrak{t}^\nu_\eta} T_w$ for some $w \in \mathfrak{S}_n \times \mathfrak{S}_\gamma$. By Lemma 2.4 and Proposition 2.5 in $S^\nu_{\mathcal{Z}}$ we have

$$m_\mathfrak{t} L_{\lambda\mu} = \sum_{\mathsf{S}\in\mathcal{T}_0^\nu(\mu,\eta)} q^C \prod_{r=1}^{z-1} \Big([\mathsf{S}_r^{(r,z]}]_q^! \prod_{j=0}^{\gamma-\mathsf{S}_r^{(r,z]}-1} [c_z - c_r - j]_q \Big) m_\mathsf{S} T_w.$$

If $\gamma < e$ then, as in the proof of Theorem 2.7, there exists a tableau S with coefficient $\zeta^C[\gamma]_\zeta^{z-1} \neq 0$ when we specialize at $q = \zeta$. Therefore, in this case we set $q = \zeta$ and argue exactly as in the proof of Theorem 2.7 to show that multiplication by $L_{\lambda\mu}$ induces a non-zero homomorphism in $\mathscr{H}om_{\mathscr{H}_n}(S\lambda, S^\mu) \neq 0$. If $\gamma \geq e$ then we have to work harder because the coefficients on the right hand side are almost always zero when we specialize to $\mathscr{H}_{n+\gamma}$.

Suppose $1 \leq r < z$. By Lemma 3.24 below, there exists an integer β_r with $0 \leq \beta_r \leq \gamma$ such that for all integers δ with $0 \leq \delta \leq \gamma$ there exist polynomials $f_{r,\delta}(q)$ and $g_{r,\delta}(q)$ in $\mathcal{Z}$, which depend only on $c_z - c_r$, such that $g_{r,\delta}(\zeta) \neq 0$ and

$$\frac{[\delta]! \prod_{j=0}^{\gamma-\delta-1}[c_z - c_r - j]_q}{[\beta_r]! \prod_{j=0}^{\gamma-\beta_r-1}[c_z - c_r - j]_q} = \frac{f_{r,\delta}(q)}{g_{r,\delta}(q)}.$$

Hence, there is a well-defined $\mathscr{H}_n^{\mathcal{Z}}$-module homomorphism $\theta^{\mathcal{Z}}_{\lambda\mu} \in \mathrm{End}_{\mathscr{H}_n}(S^\nu_{\mathcal{Z}})$ given by $\theta^{\mathcal{Z}}_{\lambda\mu}(h) = \frac{1}{\beta_{\lambda\mu}} h L_{\lambda\mu}$, for all $h \in S^\nu_{\mathcal{Z}}$, where

$$\beta_{\lambda\mu} = \beta_{\lambda\mu}(q) = \prod_{r=1}^{z-1} \left([\beta_r]_q^! \prod_{j=0}^{\gamma-\beta_r-1} [c_z - c_r - j]_q \right) \cdot \prod_{\delta=0}^{\gamma} \prod_{r=1}^{z-1} \frac{1}{g_{r,\delta}(q)}.$$

Since λ and μ form an (e, p)-Carter-Payne pair, we have $c_z - c_1 \equiv 0 \pmod{ep^{\ell_p(\gamma^*)}}$, where $\gamma^* = \lfloor \frac{\gamma}{e} \rfloor$. Consequently, by Lemma 3.24, $\beta_1 = \gamma$ and $f_{1,\delta}(\zeta) \neq 0$ if and only if $\delta = \beta_1$. Therefore, arguing as in the proof of Theorem 2.7, we see that specializing at $q = \zeta$ gives a $\mathscr{H}_n$-module homomorphism $\theta_{\lambda\mu}\colon S^\lambda \longrightarrow S^\mu$ such that

$$\pi_\lambda\theta_{\lambda\mu} = \sum_{\mathsf{S}\in\mathcal{T}_0^\nu(\mu,\lambda)} \left(\zeta^C \prod_{r=1}^{z-1} f_{r,\mathsf{S}_r^{(r,z]}}(\zeta) \prod_{\delta=0}^{\gamma} g_{r,\delta}(\zeta) \right) \varphi_\mathsf{S}.$$

Finally, to show that $\theta_{\lambda\mu}$ is non-zero we show that there exists a tableau $\mathsf{S} \in \mathcal{T}_0^\nu(\mu, \lambda)$ such that $\mathsf{S}_r^{(r,z]} = \beta_r$, for $1 \leq r < z$. This is enough because for such a tableau S the

paragraphs above show that m_{S} appears in $\theta_{\lambda\mu}(m_{\mathfrak{t}_\lambda^\nu})$ with coefficient $\prod_{r=1}^{z-1}\prod_{\delta=0}^{\gamma} g_{r,\delta}(\zeta)$, and this is non-zero by construction.

In general, there are many tableaux $\mathsf{S}\in\mathcal{T}_0^\nu(\mu,\lambda)$ with $\mathsf{S}_r^{(r,z]}=\beta_r$, for $1\le r<z$. To construct a family of tableaux with this property set $\beta_z=\gamma$. For $1\le r\le z$ we construct a partition $\nu^{(r)}$ and a semistandard $\nu^{(r)}$-tableaux $\mathsf{S}^{(r)}$ of type $(\lambda_1,\dots,\lambda_r)$ with the properties that $(\mathsf{S}^{(r)})_k^k=\nu_k-\beta_k$, for $1\le k\le r$, and

$$(\dagger)\qquad \nu_1^{(r)}+\dots+\nu_r^{(r)}=\nu_1+\dots+\nu_r-\gamma.$$

To start, let $\mathsf{S}^{(1)}$ be the unique semistandard (λ_1)-tableau of type (λ_1). By induction we may assume that we have constructed a semistandard $\nu^{(r)}$-tableau $\mathsf{S}^{(r)}$ as above. Now define $\mathsf{S}^{(r+1)}$ to be any $\nu^{(r+1)}$-tableau of type $(\lambda_1,\dots,\lambda_{r+1})$ which is obtained by adding λ_{r+1} entries labeled $r+1$ to $\mathsf{S}^{(r)}$ in such a way that $\nu^{(r+1)}\subset\nu$ and $\nu_{r+1}^{(r+1)}=\nu_{r+1}-\beta_{r+1}$. Such tableaux exist because of ($\dagger$) since $\beta_{r+1}\le\gamma\le\nu_{r+1}$. The tableau $\mathsf{S}^{(r+1)}$ is semistandard because $\lambda_i-\lambda_{i+1}\ge\gamma$, for $1\le i\le r$. It is easy to check that $\mathsf{S}^{(r+1)}$ satisfies all of the properties that we assumed of $\mathsf{S}^{(r)}$, so proceeding in this way we can construct a semistandard $\nu^{(z)}$-tableaux of type $\lambda=(\lambda_1,\dots,\lambda_z)$. In fact, $\nu^{(z)}=\mu$ by ($\dagger$) because, by construction, $(\mathsf{S}^{(z)})_z^z=\nu_z-\beta_z=\mu_z$ since $\beta_z=\gamma$. Therefore, if we define $\mathsf{S}=\mathsf{S}^{(z+1)}$ to be the tableau obtained by adding entries labeled $z+1,\dots,z+\gamma$ in row order to row z of $\mathsf{S}^{(z)}$ then $\mathsf{S}\in\mathcal{T}_0^\nu(\mu,\lambda)$ and $\mathsf{S}_r^{(r,z]}=\beta_r$, for $1\le r<z$. Consequently, the coefficient of m_{S} in $\theta_{\lambda\mu}(m_{\mathfrak{t}_\lambda^\nu})$ is non-zero, so $\theta_{\lambda\mu}\ne 0$ as claimed. □

If $p>0$ let $\beta_{\lambda\mu}(q)\in\mathbb{Z}[q]$ be the polynomial defined during the proof of Theorem 2.7 and if $p=0$ set $\beta_{\lambda\mu}(q)=1$. Then the Carter-Payne homomorphisms $\theta_{\lambda\mu}\colon S^\lambda\longrightarrow S^\mu$ that we constructed in the proofs of Theorem 2.7 and Theorem 2.8 are both of the form

$$\theta_{\lambda\mu}(m_{\mathfrak{t}})=\frac{1}{\beta_{\lambda\mu}(\zeta)}m_{\mathfrak{t}}L_{\lambda\mu}, \tag{2.9}$$

for $\mathfrak{t}\in\operatorname{Std}_\lambda(\nu)$ (and this expression makes sense).

2.10. **Example** As in Example 2.6, suppose that $\lambda=(4,4,2)$ and $\mu=(6,4)$. Then λ and μ form an (e,p)-Carter-Payne pair with $e=2$ and $p=3$. Dividing all of the equations in Example 2.6 by $[2]=1+q$ we obtain a map $\theta_{\lambda\mu}\colon S^{(4,4,2)}\longrightarrow S^{(6,4)}$. In fact, the calculations in Example 2.6 show that $\pi_\lambda\theta_{\lambda\mu}=\varphi_{\mathsf{S}}$ where

$$\mathsf{S}=\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&2&3\\\hline 2&2&2&3\\\cline{1-4}\end{array}\,.$$

However, applying Lemmas 5 and 7 from [**12**, §2] it is possible to show that if $e=p=2$ then

$$\dim\operatorname{Hom}_{\mathscr{H}_{10}}(S^{(4,4,2)},S^{(6,4)})=1.$$

The existence of such a map is not predicted by the Carter-Payne theorem. Moreover, looking at Example 2.6 shows that this map is not induced by right multiplication by any multiple of $L_{\lambda\mu}$ because in order to make this map non-zero we need to divide by $[2]_\zeta$ but then

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&4\\\hline 2&2&2&5\\\cline{1-4} 3&3\\\cline{1-2}\end{array}\;\frac{1}{[2]_\zeta}L_{\lambda\mu}=\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&3\\\hline 2&2&2&3\\\cline{1-4} 4&5\\\cline{1-2}\end{array}\ne 0,$$

when we set $\zeta=-1$. Consequently, right multiplication by $L_{\lambda\mu}/[2]_\zeta$ does not induce a homomorphism from S^λ to S^μ when $e=p=2$ because, using the notation of Lemma 2.2, the submodule M_1 of S^ν is not killed by $L_{\lambda\mu}$. ♢

2.5. Composing Homomorphisms.

This section shows that we can compose certain Carter-Payne homomorphisms. This gives a positive answer to a question of Henning Andersen.

Recall that Theorems 2.7 and 2.8 construct a non-zero homomorphism $\theta_{\lambda\sigma}\colon S^\lambda\longrightarrow S^\sigma$ whenever λ and σ form a separated Carter-Payne pair with parameters (a,y,γ). Let μ be another partition of n and suppose that $a<y<z$. Then it is easy to see that λ and μ form a separated Carter-Payne pair with parameters (a,z,γ) if and only if σ and μ form a

separated Carter-Payne pair with parameters (y,z,γ). Thus we have two homomorphisms $\theta_{\lambda\mu}$ and $\theta_{\lambda\sigma}\theta_{\sigma\mu}$, which may be the zero map, from S^λ to S^μ.

Theorem 2.11. *Suppose that λ, μ and σ are partitions of n such that λ and σ form a separated (e,p)-Carter-Payne pair with parameters (a,y,γ) and that σ and μ form a separated (e,p)-Carter-Payne pair with parameters (y,z,γ), where $a<y<z$ and $\gamma>0$. Then $\theta_{\lambda\mu}=\theta_{\lambda\sigma}\theta_{\sigma\mu}$.*

Proof. Using Proposition 2.1, we may assume that $a=1$ and $z=\max\{\,i>0\mid\lambda_i\neq0\,\}$. Let ν be the partition of $n+\gamma$ given by

$$\nu_i=\begin{cases}\lambda_i+\gamma, & \text{if } i=1,\\ \lambda_i, & \text{otherwise.}\end{cases}$$

Then $\lambda,\mu,\sigma\subset\nu$.

To prove the Theorem we consider the Specht module $S^\nu_{\mathcal Z}$ for the generic Iwahori-Hecke algebra $\mathscr H^{\mathcal Z}_{n+\gamma}$. As in Lemma 2.2, we fix a Specht filtration

$$S^\nu_{\mathcal Z}=M_0\supset M_1\supset\cdots\supset M_l\supset\cdots\supset M_k\supset 0$$

of $S^\nu_{\mathcal Z}$ such that, as $\mathscr H^{\mathcal Z}_n$-modules, $S^\lambda_{\mathcal Z}\cong M_0/M_1$, $S^\mu_{\mathcal Z}\cong M_k$ and $S^\sigma_{\mathcal Z}\cong M_l/M_{l+1}$ for some $1\le l<k$. We may assume that $\{\,m_{\mathfrak t}\mid\sigma\not\trianglerighteq\operatorname{Shape}(\mathfrak t_{\downarrow n})\,\}$ is a basis of M_{l+1}. For $1\le i\le z$, set $c_i=\nu_i-i$. Mirroring the definition of $L_{\lambda\mu}$ (see before Lemma 2.4), set

$$L_{\lambda\sigma}=\prod_{i=1}^{y-1}\prod_{j=1}^{\gamma}(L_{n+j}-[c_i]_q)\qquad\text{and}\qquad L_{\sigma\mu}=\prod_{i=y}^{z-1}\prod_{j=1}^{\gamma}(L_{n+j}-[c_i]_q).$$

Then $L_{\lambda\mu}=L_{\lambda\sigma}L_{\sigma\mu}$. By (2.9) there exist polynomials $\beta_{\lambda\mu}(q),\beta_{\sigma\mu}(q)\in\mathbb Z[q]$ such that

$$\theta_{\lambda\mu}(m_{\mathfrak t})=\frac{1}{\beta_{\lambda\mu}(\zeta)}m_{\mathfrak t}L_{\lambda\mu},$$

for $\mathfrak t\in\operatorname{Std}_\lambda(\nu)$. Via Proposition 2.1, we have analogous descriptions of the maps $\theta_{\lambda\sigma}$ and $\theta_{\sigma\mu}$, however, we do not (yet) have a description of these maps as $\mathscr H_n$-module endomorphisms of S^ν. The next three claims allow us to describe these maps as endomorphisms of S^ν and to connect them with $\theta_{\lambda\mu}$.

Claim 1. *Suppose that η is a partition of n such that $\eta\subset\nu$ and $\eta\trianglerighteq\sigma$. Then, in $S^\nu_{\mathcal Z}$,*

$$m_{\mathfrak t^\nu_\eta}L_{\sigma\mu}=q^{C_1}\sum_{\mathsf S\in\mathcal T^\nu_0(\mu,\eta)}\prod_{r=y}^{z-1}\Big([\mathsf S^{(r,z]}_r]^!_q\prod_{j=0}^{\gamma-\mathsf S^{(r,z]}_r-1}[c_z-c_r-j]_q\Big)m_{\mathsf S}.$$

for some $C_1\in\mathbb Z$. Moreover, if $\mathsf S\in\mathcal T^\nu_0(\mu,\eta)$ then $\mathsf S^r_r=\mu_r$, for $1\le r\le y$.

Proof of Claim 1. When $y=1$ this is precisely Proposition 2.5. We are assuming, however, that $y>1$. In this case, the formula for $m_{\mathfrak t^\nu_\eta}L_{\lambda\sigma}$ follows by setting $k=\gamma$ in Proposition 3.19 below (which includes Proposition 2.5 as a special case). Secondly, observe that $\operatorname{row}_{\mathfrak t^\nu_\eta}(n+j)\ge y$, for $1\le j\le\gamma$, because $\eta\trianglerighteq\sigma$. Consequently, if $\mathsf S\in\mathcal T^\nu_0(\mu,\eta)$ then $\eta_r=\mathsf S^r_r=\mu_r$, for $1\le r\le y$. □

Claim 2. *Suppose that η is a partition of n such that $\eta\subset\nu$. Then, in $S^\nu_{\mathcal Z}/M_{l+1}$,*

$$m_{\mathfrak t^\nu_\eta}L_{\lambda\sigma}\equiv q^{C_2}\sum_{\mathsf S\in\mathcal T^\nu_0(\sigma,\eta)}\prod_{r=1}^{y-1}\Big([\mathsf S^{(r,y]}_r]^!_q\prod_{j=0}^{\gamma-\mathsf S^{(r,y]}_r-1}[c_y-c_r-j]_q\Big)m_{\mathsf S}\pmod{M_{l+1}}.$$

for some $C_2\in\mathbb Z$. Moreover, if $\mathsf S\in\mathcal T^\nu_0(\sigma,\eta)$ then $\mathsf S^r_r=\sigma_r$, for $y\le r\le z$.

Proof of Claim 2. First observe that, by Lemma 3.12 below, $m_{\mathfrak{t}^\nu_\eta} L_{\lambda\sigma}$ is a linear combination of terms $m_\mathfrak{s}$ where $\mathfrak{s}_{\downarrow n} \trianglerighteq \mathfrak{t}^\eta$. If $\operatorname{row}_\mathfrak{s}(n+j) > y$ for some j with $1 \le j \le \gamma$ then $m_\mathfrak{s} \in M_{l+1}$, so we may assume that $\operatorname{row}_\mathfrak{s}(n+j) \le y$ for $1 \le j \le \gamma$. Consequently, if $m_\mathsf{S} + S_{l+1}$ appears with non-zero coefficient in $m_{\mathfrak{t}^\nu_\eta} L_{\lambda\sigma}$ for some $\mathsf{S} \in \mathcal{T}_0^\nu(\mu,\eta)$ then $\eta_r = \mathsf{S}_r^r = \sigma_r$, for $y \le r \le z$. Therefore, we may replace σ with $(\sigma_1,\dots,\sigma_y)$ and deduce the claim from Proposition 2.5. Note that if $\mathsf{S} \in \mathcal{T}_0^\nu(\sigma,\eta)$ and $1 \le r < y$ then $\mathsf{S}_r^{(r,y]} = \mathsf{S}_r^{(r,z)}$ since $\mathsf{S}_a^a = \sigma_a$ when $y \le a \le z$. □

Claim 3. *Suppose that η is a partition of n such that $\eta \subset \nu$. Then*

$$m_{\mathfrak{t}^\nu_\eta} L_{\lambda\mu} \equiv q^C \sum_{\mathsf{S}\in\mathcal{T}_0^\nu(\mu,\sigma,\eta)} \prod_{r=1}^{z-1} \Big([\mathsf{S}_r^{(r,z]}]_q^! \prod_{j=0}^{\gamma-\mathsf{S}_r^{(r,z]}-1} [c_z - c_r - j]_q \Big) m_\mathsf{S} \pmod{[c_y - c_z] S^\nu_\mathcal{Z}}.$$

where $C \in \mathbb{Z}$ and $\mathcal{T}_0^\nu(\mu,\sigma,\eta) = \{\, \mathsf{S} \in \mathcal{T}_0^\nu(\mu,\eta) \mid \mathsf{S}_r^{>y} = 0$ for $1 \le r < y \,\}$.

Proof of Claim 3. Proposition 2.5 shows that $m_{\mathfrak{t}^\nu_\eta} L_{\lambda\mu}$ is a linear combination of terms m_S, for $\mathsf{S} \in \mathcal{T}_0^\nu(\mu,\eta)$ and, moreover, if $\mathsf{S} \in \mathcal{T}_0^\nu(\mu,\sigma,\eta)$ then the coefficient of m_S is exactly as above. On the other hand, if $\mathsf{S} \in \mathcal{T}_0^\nu(\mu,\eta) \setminus \mathcal{T}_0^\nu(\mu,\sigma,\eta)$ then $\mathsf{S}_y^{(y,z]} < \gamma$ so, by Proposition 2.5 again, the coefficient of m_S in $m_{\mathfrak{t}^\nu_\eta} L_{\lambda\mu}$ is divisible by $[c_y - c_z]$. This proves the claim. □

Armed with these three claims we now return to the proof of Theorem 2.11. Combining Claims 1–3 shows that if $\mathfrak{t} \in \operatorname{Std}(\nu)$ then, modulo $[c_y - c_z] S^\nu_\mathcal{Z}$, $m_\mathfrak{t} L_{\lambda\mu} = m_\mathfrak{t} L_{\lambda\sigma} L_{\sigma\mu}$ is equal to a linear combination of terms m_S where $\mathsf{S} \in \mathcal{T}_0^\nu(\mu,\sigma,\eta)$ where the coefficient of m_S is equal to the product of the coefficients coming from multiplication by $L_{\lambda\sigma}$ (Claim 2) and multiplication by $L_{\sigma\mu}$ (Claim 1). (Furthermore, $C = C_1 + C_2$.) The coefficients in Claim 1 determine the polynomials $\beta_{\sigma\mu}(q)$, via Lemma 3.24. Similarly, the coefficients in Claim 2 determine the polynomials $\beta_{\lambda\sigma}(q)$ and those in Claim 3 determine $\beta_{\lambda\mu}(q)$. By Lemma 3.24 the polynomial $\beta_{\lambda\sigma}(q)\beta_{\sigma\mu}(q)$ divides all of the coefficients of the terms appearing in $m_{\mathfrak{t}^\nu_\eta} L_{\lambda\mu}$ according to Proposition 2.5. Therefore, in the proof of Theorem 2.8 we can take $\beta_{\lambda\mu}(q) = \beta_{\lambda\sigma}(q)\beta_{\sigma\mu}(q)$. Note that, as in the proof of Theorem 2.8, the terms in $[c_y - c_z] S^\nu_\mathcal{Z}$ in Claim 3 do not contribute to the image of $\theta_{\lambda\mu}$ because $c_z - c_y \equiv 0 \pmod{e p^{\ell_p(\gamma^*)}}$. Therefore, $\theta_{\lambda\mu} = \theta_{\lambda\sigma}\theta_{\sigma\mu}$ as required. □

Remark. The polynomials $\beta_{\lambda\mu}(q) \in F[q]$ are not uniquely determined by Lemma 3.24. The proof of Theorem 2.11 really shows that we can choose these polynomials so that, under the assumptions of the theorem, $\beta_{\lambda\mu}(q) = \beta_{\lambda\sigma}(q)\beta_{\sigma\mu}(q)$. Without this choice of β-polynomials, all we can say is that $\theta_{\lambda\mu} = u\theta_{\lambda\sigma}\theta_{\sigma\mu}$ for some non-zero scalar $u \in F$.

2.6. **Jantzen filtrations.** In this section we connect the Jantzen filtrations and the Carter-Payne homomorphisms constructed in Section 2.4. If $p = 0$ our result says that the image is in contained in the radical of S^μ, which is automatically true, so this result is most interesting when F is a field of positive characteristic. The key to the proof is the observation that if $q = \zeta$ then we can write $L_{\lambda\mu}$ in two different ways using the element $L'_{\lambda\mu}$ defined below.

The Hecke algebra $\mathscr{H}_n$ is defined over the field F with parameter ζ. Let q be an indeterminate over F and let $\mathscr{O} = F[q]_{(q)}$ be the localization of $F[q]$ at the maximal ideal generated by q. Then $\mathscr{O}$ is a discrete valuation ring with maximal ideal $\pi = q\mathscr{O}$, the polynomials in $F[q]$ with zero constant term. For $0 \ne f \in \mathscr{O}$ define $\mathit{val}_\pi(f) = k$ where k is maximal such that $f \in \pi^k$. Let $K = F(q)$ be the field of fractions of $\mathscr{O}$. We consider F as an $\mathscr{O}$-module by letting q act on F as multiplication by ζ.

Let $\mathscr{H}_n^\mathscr{O}$ be the Hecke algebra of $\mathfrak{S}_n$ over $\mathscr{O}$ with (invertible) parameter $q + \zeta$. Then $\mathscr{H}_n \cong \mathscr{H}_n^\mathscr{O} \otimes_\mathscr{O} F$ and $\mathscr{H}_n^K = \mathscr{H}_n^\mathscr{O} \otimes_\mathscr{O} K$ is (split) semisimple. Thus $(K, \mathscr{O}, F)$ is a modular system, with parameter $q + \zeta$, for the algebras $(\mathscr{H}_n^K, \mathscr{H}_n^\mathscr{O}, \mathscr{H}_n)$.

The algebra $\mathscr{H}_n^{\mathscr{O}}$ is cellular with cell modules the Specht modules $S^\mu_{\mathscr{O}}$, for μ a partition of n. We have that $S^\mu_K = S^\mu_{\mathscr{O}} \otimes_{\mathscr{O}} K$ is irreducible and $S^\mu = S^\mu_F = S^\mu_{\mathscr{O}} \otimes_{\mathscr{O}} F$ is the $\mathscr{H}_n$-module defined in section 2.1. As $\mathscr{H}_n^{\mathscr{O}}$ is cellular, the Specht module $S^\mu_{\mathscr{O}}$ comes equipped with a bilinear form $\langle\ ,\ \rangle_{\mathscr{O},\mu} = \langle\ ,\ \rangle_\mu$. For each positive integer i define

$$J^i(S^\mu_{\mathscr{O}}) = \{\, x \in S^\mu_{\mathscr{O}} \mid \langle x, y\rangle_\mu \in \pi^i \text{ for all } y \in S^\mu_{\mathscr{O}} \,\}.$$

Finally, define $J^i(S^\mu) = (J^i(S^\mu_{\mathscr{O}}) + \pi J^i(S^\mu_{\mathscr{O}}))/J^i(S^\mu_{\mathscr{O}})$, for $i \in \mathbb{Z}$. Then

$$S^\mu = J^0(S^\mu) \supseteq J^1(S^\mu) \supseteq \dots$$

is the **Jantzen filtration** of S^μ relative to the modular system $(K, \mathscr{O}, F)$.

As in the last section, we assume that λ and μ form a separated (e,p)-Carter-Payne pair with parameters (a, z, γ). We can again assume that $a = 1$, $z = \max\{\, i \mid \lambda_i \neq 0 \,\}$ and we define ν to be the partition of $n+\gamma$ obtained by adding γ nodes to the first row of λ.

As a slight variation on the definition of $L_{\lambda\mu}$ in section 2.2 set

$$L'_{\lambda\mu} = \prod_{i=1}^{z-1}\prod_{j=1}^{\gamma-1}(L_{n+j} - [c_i]) \cdot \prod_{i=2}^{z}(L_{n+\gamma} - [c_i]).$$

Since $\prod_{i=2}^{z}(L_{n+\gamma} - [c_i])$ divides $L'_{\lambda\mu}$, the following result is easily proved using Lemma 3.12 below. We leave it as an exercise for the reader.

Lemma 2.12. *Suppose that* $\mathfrak{t} \in \operatorname{Std}(\nu)$ *and that* $\operatorname{row}_{\mathfrak{t}}(n+\gamma) > 1$. *Then* $m_{\mathfrak{t}} L'_{\lambda\mu} = 0$.

As a consequence, if M_1 is the submodule of S^ν which appears in the filtration of S^ν described in Lemma 2.2, then $M_1 L'_{\lambda\mu} = 0$.

The Specht module $S^\nu_{\mathscr{O}}$ also carries an analogous inner product $\langle\ ,\ \rangle_\nu$. The inner products $\langle\ ,\ \rangle_\mu$ and $\langle\ ,\ \rangle_\nu$ are determined by the multiplication in $\mathscr{H}_n$ and $\mathscr{H}_{n+\gamma}$, respectively; see, for example, [**17**, (2.8)]. These inner products are associative in the sense that $\langle xh, y\rangle_\nu = \langle x, yh^*\rangle$, for all $x, y, \in S^\nu_{\mathscr{O}}$ and $h \in \mathscr{H}^{\mathscr{O}}_{n+\gamma}$, where $*$ is the unique anti-isomorphism of $\mathscr{H}^{\mathscr{O}}_{n+\gamma}$ such that $T_w^* = T_{w^{-1}}$ for all $w \in \mathfrak{S}_{n+\gamma}$. In particular, if $1 \le k \le n+\gamma$ then $\langle xL_k, y\rangle_\nu = \langle x, yL_k\rangle_\nu$, so that $\langle xL_{\lambda\mu}, y\rangle_\nu = \langle x, yL_{\lambda\mu}\rangle_\nu$, for all $x, y, \in S^\nu_{\mathscr{O}}$. Proofs of all of these facts can be found in [**17**, Chapt. 2].

Since $\mathfrak{t}^\nu_\mu = \mathfrak{t}^\nu$ we have the following.

Lemma 2.13. *Consider* $S^\mu_{\mathscr{O}}$ *as an* $\mathscr{H}_n$*-submodule of* $S^\nu_{\mathscr{O}}$ *as in Lemma 2.2. Then*

$$\langle x, y\rangle_\nu = \langle x, y\rangle_\mu, \qquad \textit{for all } x, y \in S^\mu_{\mathscr{O}}.$$

Recall that we defined the map $\mathscr{val}_{e,p}$ just before the statement of Theorem 1.3 in the introduction and that (2.9) defines a polynomial $\beta_{\lambda\mu}(q) \in F[q]$ whenever λ and μ form a Carter-Payne pair.

We can now prove Theorem 1.3 from the introduction.

Proof of Theorem 1.3. We have to show that the image of $\theta_{\lambda\mu}$ is contained in $J^\delta(S^\mu)$, where $\delta = \mathscr{val}_{e,p}(\lambda_a - \lambda_z + z - a + \gamma) - \mathscr{val}_{e,p}(\gamma)$. To do this we work in $\mathscr{H}^{\mathscr{O}}_{n+\gamma}$. Let $L^{\mathscr{O}}_{\lambda\mu}$ and $L'^{\mathscr{O}}_{\lambda\mu}$ be the elements of $\mathscr{H}^{\mathscr{O}}_n$ which are obtained from $L_{\lambda\mu}$ and $L'_{\lambda\mu}$, respectively, by replacing q with $q+\zeta$. Using the simple identity $[c_1]_{q+\zeta} = [c_z]_{q+\zeta} + q^{c_z}[c_1 - c_z]_{q+\zeta}$, we see that

$$L^{\mathscr{O}}_{\lambda\mu} = \prod_{i=1}^{z-1}\prod_{j=1}^{\gamma}(L_{n+j} - [c_i]_{q+\zeta}) = L'^{\mathscr{O}}_{\lambda\mu} - q^{c_z}[c_1 - c_z]_{q+\zeta} L''^{\mathscr{O}}_{\lambda\mu},$$

where $L''^{\mathscr{O}}_{\lambda\mu} = \prod_{i=2}^{z-1}\prod_{j=1}^{\gamma}(L_{n+j} - [c_i]_{q+\zeta}) \cdot \prod_{j=1}^{\gamma-1}(L_{n+j} - [c_1]_{q+\zeta})$. Therefore, when we specialize at $q = 0$,

$$L_{\lambda\mu} = L^{\mathscr{O}}_{\lambda\mu} \otimes_{\mathscr{O}} 1 = L'^{\mathscr{O}}_{\lambda\mu} \otimes_{\mathscr{O}} 1 = L'_{\lambda\mu}$$

in $\mathscr{H}_n$ since $c_1 \equiv c_z \pmod e$. So multiplication by $L_{\lambda\mu}$ and $L'_{\lambda\mu}$ induce the same $\mathscr{H}_n$-homomorphism $S^\lambda \to S^\mu$, which may be zero, by the argument of Theorem 2.7.

In the proof of Theorem 2.8, the homomorphism $\theta_{\lambda\mu}$ was defined to be the specialization of the map $m_\mathfrak{t} \mapsto \frac{1}{\beta_{\lambda\mu}(q+\zeta)} m_\mathfrak{t} L^{\mathscr{O}}_{\lambda\mu}$ at $q = 0$, for $\mathfrak{t} \in \operatorname{Std}_\lambda(\nu)$. Set $h = \lambda_a - \lambda_z + z - a + \gamma = c_1 - c_z$, so that $\delta = \mathit{val}_{e,p}(h)$. By assumption, if $l = \ell_p(\gamma^*)$ then $h \equiv 0 \pmod{ep^l}$. If we write $h = h'ep^l$, for some $h' \in \mathbb{Z}$, then

$$[c_1 - c_z]_{q+\zeta} = [h'ep^l]_{q+\zeta} = [ep^l]_{q+\zeta}[h']_{(q+\zeta)^{p^l}} = [e]^{p^l}[h']_{(q+\zeta)^{p^l}}.$$

Hence, $\mathit{val}_\pi([h]_{q+\zeta}) \geq p^l = \mathit{val}_{e,p}(h) = \delta$.

Recall that $L^{\mathscr{O}}_{\lambda\mu} = L'^{\mathscr{O}}_{\lambda\mu} + [c_1 - c_z]_{q+\zeta} L''^{\mathscr{O}}_{\lambda\mu}$. Suppose that $\mathfrak{t} \in \operatorname{Std}_\lambda(\nu)$. By Lemma 2.13, if x belongs to $S^\mu_{\mathscr{O}}$ then

$$\begin{aligned}\langle m_\mathfrak{t} L^{\mathscr{O}}_{\lambda\mu}, x\rangle_\mu = \langle m_\mathfrak{t} L^{\mathscr{O}}_{\lambda\mu}, x\rangle_\nu &= \langle m_\mathfrak{t} L'^{\mathscr{O}}_{\lambda\mu}, x\rangle_\nu - q^{c_z}[h]_{q+\zeta}\langle m_\mathfrak{t} L''^{\mathscr{O}}_{\lambda\mu}, x\rangle_\nu \\ &= -q^{c_z}[h]_{q+\zeta}\langle m_\mathfrak{t} L''^{\mathscr{O}}_{\lambda\mu}, x\rangle_\nu,\end{aligned}$$

where the last equality follows because $\langle m_\mathfrak{t} L'^{\mathscr{O}}_{\lambda\mu}, x\rangle_\nu = \langle m_\mathfrak{t}, x L'^{\mathscr{O}}_{\lambda\mu}\rangle_\nu = 0$ by Lemma 2.12.

If $\gamma < e$ then $\beta_{\lambda\mu}(q+\zeta) = 1$ and the proof is complete. If $\gamma \geq e$ it remains to account for dividing by $\beta_{\lambda\mu}(q+\zeta)$ in the definition of $\theta_{\lambda\mu}$. Observe that if $x \in S^\mu_{\mathscr{O}}$ then x is a linear combination of terms $m_\mathfrak{s}$ with $\mathfrak{s} \in \operatorname{Std}_\mu(\nu)$. If $\mathfrak{s} \in \operatorname{Std}_\mu(\nu)$ then $\operatorname{row}_\mathfrak{s}(n+j) = z$, for $1 \leq j \leq \gamma$. Therefore, $m_\mathfrak{s} L_{n+j} = [c_z - j + 1] m_\mathfrak{s}$ by Lemma 3.12 below, for example, so that

$$\begin{aligned}\langle m_\mathfrak{t} L''^{\mathscr{O}}_{\lambda\mu}, x\rangle_\nu &= \langle m_\mathfrak{t}, x L''^{\mathscr{O}}_{\lambda\mu}\rangle_\nu \\ &= \prod_{i=2}^{z-1}\prod_{j=0}^{\gamma-1} q^{c_i}[c_z - c_i - j]_{q+\zeta} \cdot \prod_{j=0}^{\gamma-2} q^{c_1}[c_z - c_1 - j]_{q+\zeta} \cdot \langle m_\mathfrak{t}, x\rangle_\nu.\end{aligned}$$

Let $\beta'_{\lambda\mu}(q+\zeta)$ be the coefficient of $\langle m_\mathfrak{t}, x\rangle$ in the last equation. Recall from the proof of Theorem 2.8 that the polynomial $\beta_{\lambda\mu}(q+\zeta)$ is a product of $z-1$ factors corresponding to the row index $i = 1, 2, \dots, z-1$ above. Noting that $c_1 \equiv c_z \pmod e$, we have that

$$\mathit{val}_\pi\big([\gamma]_{q+\zeta}\beta'_{\lambda\mu}(q+\zeta)\big) \geq \mathit{val}_\pi\big(\beta_{\lambda\mu}(q+\zeta)\big)$$

by taking $X = 0$ in Corollary 3.23. This completes the proof. $\square$

It would be interesting to know how tight the bound obtained in Theorem 1.3 is. That is, to determine the maximal δ' such that the image of $\theta_{\lambda\mu}$ is contained in $J^{\delta'}(S^\mu)$.

If $\gamma < e$ then $\beta_{\lambda\mu}(q) = 1$. Hence, as a special case of the Theorem we obtain the following.

Corollary 2.14. *Suppose that $p > 0$, $\gamma < e$ and that λ and μ form an (e,p)-Carter-Payne pair with parameters (a, z, γ) such that $\lambda_r - \lambda_{r+1} \geq \gamma$, whenever $a \leq r \leq z$. Then $\operatorname{Im}\theta_{\lambda\mu} \subseteq J^\delta(S^\mu)$, where $\delta = \mathit{val}_{e,p}(\lambda_a - \lambda_z + z - a + \gamma)$.*

When $\zeta = 1$ and $\gamma = 1$ this result has already been proved by Ellers and Murray [**11**, Theorem 7.1] without assuming that $\lambda_r - \lambda_{r+1} \geq \gamma$, for $a \leq r \leq z$. The proof of Theorem 1.3 was inspired by the argument of Ellers and Murray.

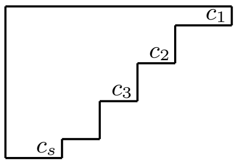
We note that when $\zeta = 1$ we can replace the modular system $(K, \mathscr{O}, F)$ used above with $(\mathbb{Q}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}/p\mathbb{Z})$ and the valuation map val_π with the usual p-adic valuation map val_p. With these choices, we obtain the 'natural' Jantzen filtration of S^μ and the argument above shows that we can take $\delta = \mathit{val}_p(c_1 - c_z) - \mathit{val}_p(\gamma)$.

2.7. **The (e,p)-Carter-Payne Theorem.** The techniques used in this paper to prove Theorem 2.7 and Theorem 2.8 can be used to prove the existence of homomorphisms between other pairs of Specht modules. As we now sketch, it is likely that a complete proof of Theorem 1.1 could be given using these ideas.

Fix a pair of partitions λ and μ of n which form a Carter-Payne pair with parameters (a, z, γ). As in the last section we may assume that $a = 1$ and that z is the length of λ. Let ν be the partition of $n + \gamma$ given by

$$\nu_r = \begin{cases} \lambda_r + \gamma, & r = 1, \\ \lambda_r, & \text{otherwise.} \end{cases}$$

Write $\nu = (\nu_1^{b_1}, \nu_2^{b_2}, \dots, \nu_s^{b_s})$ where $\nu_1 > \nu_2 > \dots > \nu_s > 0$, and set $B_i = \sum_{k=1}^{i} b_k$ for $1 \leq i \leq s$. Then the nodes that can be removed from $\mathbb{D}(\nu)$ to leave the diagram of a partition are at the ends of the rows $B_1, B_2, \dots, B_s$. Set $c_r = \nu_r - B_r$, for $1 \leq r \leq s$, so that c_r is the content of the r^{th} removable node of ν:

$\mathbb{D}(\nu) =$ 

Now define

$$L_{\lambda\mu} = \prod_{r=1}^{s-1} \prod_{j=1}^{\gamma} \Big(L_{n+j} - [c_r] \Big).$$

Arguing as in the proof of Theorem 2.7 or Theorem 2.8 it is possible to show that right multiplication by $L_{\lambda\mu}$ induces a $\mathscr{H}_n$-homomorphism $S^\lambda \to S^\mu$. However, it is not clear that this homomorphism is non-zero.

If λ and μ form an (e, p)-Carter-Payne pair with parameters (a, z, γ) where $\gamma = 1$ then using Corollary 3.18 below, or by following Ellers and Murray [**10**], it is possible to show that right multiplication by $L_{\lambda\mu}$ induces a non-zero $\mathscr{H}_n$-homomorphism from S^λ to S^μ.

Conjecture 2.15. Suppose that $\gamma < e$. Then right multiplication by $L_{\lambda\mu}$ induces a non-zero $\mathscr{H}_n$-homomorphism from S^λ to S^μ.

By the argument used to prove Theorem 1.3, if this conjecture is true then the image of this homomorphism is contained in $J^\delta(S^\mu)$, where $\delta = \mathcal{val}_{e,p}(\lambda_a - \lambda_z + z - a + \gamma)$.

We end with two examples.

2.16. **Example** Suppose that $\lambda = (4, 4, 3, 2)$, that $\mu = (6, 4, 3)$ and that $e = 7$. If we take $\mathfrak{t} = \mathfrak{t}_\lambda^\nu$ and $L_{\lambda\mu} = (L_{15} - [5])(L_{15} - [2])L_{15}(L_{14} - [5])(L_{14} - [2])L_{14}$ then direct computation shows that

$$\begin{aligned}
m_\mathfrak{t} L_{\lambda\mu} = {} & q^{-5}(q^3 - q - 1)[2][2][4] \begin{smallmatrix} 1&1&1&1&2&2 \\ 2&2&3&3 \\ 3&4&4 \\ 5&6 \end{smallmatrix} - q^{-5}[2][2][2][4] \begin{smallmatrix} 1&1&1&1&2&2 \\ 2&2&3&4 \\ 3&3&4 \\ 5&6 \end{smallmatrix} \\
& - q^{-4}[2][2][4] \begin{smallmatrix} 1&1&1&1&2&3 \\ 2&2&2&3 \\ 3&4&4 \\ 5&6 \end{smallmatrix} + q^{-6}[2][2][4] \begin{smallmatrix} 1&1&1&1&2&3 \\ 2&2&2&4 \\ 3&3&4 \\ 5&6 \end{smallmatrix} \\
& + q^{-6}[2][2][4] \begin{smallmatrix} 1&1&1&1&2&4 \\ 2&2&2&3 \\ 3&3&4 \\ 5&6 \end{smallmatrix} - q^{-9}[2][2][3][4] \begin{smallmatrix} 1&1&1&1&2&4 \\ 2&2&2&4 \\ 3&3&3 \\ 5&6 \end{smallmatrix} \\
& + q^{-9}[2][2][4][5] \begin{smallmatrix} 1&1&1&1&3&3 \\ 2&2&2&2 \\ 3&4&4 \\ 5&6 \end{smallmatrix} - q^{-11}[2][2][4][5] \begin{smallmatrix} 1&1&1&1&3&4 \\ 2&2&2&2 \\ 3&3&4 \\ 5&6 \end{smallmatrix} \\
& + q^{-4}[2][2][3][4][5] \begin{smallmatrix} 1&1&1&1&4&4 \\ 2&2&2&2 \\ 3&3&3 \\ 5&6 \end{smallmatrix}.
\end{aligned}$$

Further, if $\mathfrak{t} = \mathfrak{t}_\eta^\nu$ for some $\nu \neq \eta$ then $m_\mathfrak{t} L_{\lambda\mu}$ has a factor of $[7]$. Thus if $e = 7$ (and p is arbitrary) there exists a non-zero homomorphism $\theta : S^\lambda \to S^\mu$.

Note that the coefficient of the first tableau is not a product of Gaussian polynomials multiplied by a power of q. This indicates that the polynomial coefficients appearing in a general version of Proposition 2.5 may be difficult to describe. $\diamondsuit$

2.17. **Example** Finally let us consider the case that $\lambda = (4,3,3)$ and $\mu = (7,3)$. If we take $\mathfrak{t} = \mathfrak{t}^\nu_\lambda$, and $L_{\lambda\mu} = (L_{10} - [6])(L_9 - [6])(L_8 - [6])$ then direct computation shows that

$$\begin{aligned} m_\mathfrak{t} L_{\lambda\mu} = &- q^6[2][3]\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&2\\ \hline 3&3&3&\multicolumn{4}{c}{}\\ \cline{1-3} 4&4&4&\multicolumn{4}{c}{}\\ \cline{1-3}\end{array} + q^5[2]\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3\\ \hline 2&3&3&\multicolumn{4}{c}{}\\ \cline{1-3} 4&4&4&\multicolumn{4}{c}{}\\ \cline{1-3}\end{array} \\ &- q^3[2]\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&3&3\\ \hline 2&2&3&\multicolumn{4}{c}{}\\ \cline{1-3} 4&4&4&\multicolumn{4}{c}{}\\ \cline{1-3}\end{array} + [2][2]\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&3&3&3\\ \hline 2&2&2&\multicolumn{4}{c}{}\\ \cline{1-3} 4&4&4&\multicolumn{4}{c}{}\\ \cline{1-3}\end{array}. \end{aligned}$$

If $\mathfrak{t} = \mathfrak{t}^\nu_\eta$ for some $\nu \neq \eta$ then $m_\mathfrak{t} L_{\lambda\mu}$ has a factor of $[6]$. So if $e = 2$ and $p = 3$ then (after dividing by $[2]$), we have shown that there is a non-zero homomorphism between S^λ and S^μ, as predicted by the Carter-Payne theorem. However, we have shown that if $e = 3$ and p is arbitrary then we there is a non-zero homomorphism. These maps are not Carter-Payne homomorphism except when $p = 2$, although they are described by Parker [**19**].

It is interesting to note that in [**12**] the authors show the existence of such a homomorphism in the case when $e = p = 3$; that is, when $\mathscr{H}_n = F_3\mathfrak{S}_{10}$. $\diamondsuit$

3. Jucys-Murphy elements acting on almost initial tableaux

In this section we complete the proof of our main results in Sections 2.4–2.6 by proving some very precise formulas which describe how the Jucys-Murphy elements act on certain elements of the Specht modules. The results in this section are valid for an arbitrary Hecke algebra $\mathscr{H}_{n+\gamma} = \mathscr{H}^F_{n+\gamma}$ defined over an ring F with invertible parameter q. Nonetheless, throughout we work with the generic Hecke algebra $\mathscr{H}^\mathcal{Z}_{n+\gamma}$ as we prefer to think of $[k] = [k]_q$ as a polynomial in q. The results in this section are independent of the results in Sections 2.4–2.6.

Throughout this section we fix integers $n, \gamma > 0$ and an *arbitrary* partition ν of $n+\gamma$. (In this section the only result which requires the assumption that $\nu_i - \nu_{i+1} \geq \gamma$, for $1 \leq i < z$, is Proposition 3.19.) Let $z = \max\{\, r \mid \nu_r > 0\,\}$. Recall that $\{T_w \mid w \in \mathfrak{S}_{n+\gamma}\}$ is a basis of $\mathscr{H}^\mathcal{Z}_{n+\gamma}$.

3.1. **Semistandard basis elements.** We now fix notation that will be used extensively for the rest of the paper. Suppose that i and j are integers such that $1 \leq i \leq j \leq n+\gamma$. Define

$$T_{i,j} = \prod_{l=i}^{j-1} T_l$$

and for $i < k \leq j$ define

$$T_{i,j\backslash k} = \prod_{l=i}^{k-2} T_l \cdot \prod_{l=k}^{j-1} T_l.$$

Our convention will always be to read products from left to right, so that

$$T_{i,j} = T_i T_{i+1} \dots T_{j-1} \qquad \text{and} \qquad T_{i,j\backslash k} = T_i T_{i+1} \dots T_{k-2} T_k \dots T_{j-1}.$$

In particular, $T_{i,i} = 1$, $T_i = T_{i,i+1}$, $T_{i+1,j} = T_{i,j\backslash i+1}$ and $T_{i,j-1} = T_{i,j\backslash j}$. Recall that for $1 \leq k \leq n+\gamma$ we defined the Jucys-Murphy element L_k. Similarly, we set

$$L'_k = q^{1-k} T_{k-1} \dots T_1 T_{1,k}, \qquad \text{for } 1 \leq k \leq n.$$

The reader can check that $L'_k = (q-1)L_k + 1$. Consequently, the elements L_k and L'_k are almost interchangeable.

Let S^ν be the $\mathscr{H}^\mathcal{Z}_{n+\gamma}$-module corresponding to the partition ν, so that S^ν has basis $\{m_\mathfrak{t} \mid \mathfrak{t} \in \operatorname{Std}(\nu)\}$. If $\mathfrak{s} \in \operatorname{RStd}(\nu)$ and $1 \leq k \leq n$ then the **content** of k in $\mathfrak{s}$ is $\mathrm{c}_\mathfrak{s}(k) = c - r$, if $\mathfrak{s}(r,c) = k$.

Lemma 3.1. *Suppose that* $1 \le i \le n+\gamma-1$ *and that* $\mathfrak{s} \in \mathrm{RStd}(\nu)$*. Then*

$$m_{\mathfrak{s}}T_i = \begin{cases} m_{\mathfrak{s}(i,i+1)}, & i \text{ lies above } i+1 \text{ in } \mathfrak{s},\\ qm_{\mathfrak{s}}, & i \text{ and } i+1 \text{ lie in the same row of } \mathfrak{s},\\ qm_{\mathfrak{s}(i,i+1)} + (q-1)m_{\mathfrak{s}}, & \text{otherwise}. \end{cases}$$

Note that if $\mathfrak{s}$ is standard then the tableau $\mathfrak{s}(i,i+1)$ is also standard unless i and $i+1$ are in the same column.

Proof. The result holds for the row standard basis, $\{\, m_\nu T_{d(\mathfrak{s})} \mid \mathfrak{s} \in \mathrm{RStd}(\nu)\,\}$, of the permutation module $M^\nu = m_\nu \mathscr{H}^{\mathcal{Z}}_{n+\gamma}$ by [**17**, Corollary 3.4]. As $m_{\mathfrak{s}}$ is just the image of $m_\nu T_{d(\mathfrak{s})}$ under the natural projection map $M^\nu \to S^\nu$ the result follows. □

Lemma 3.2. *Suppose that* $1 \le k \le n$*. Then*

$$m_{\mathfrak{t}^\nu} L_k = [\mathrm{c}_{\mathfrak{t}^\nu}(k)] m_{\mathfrak{t}^\nu} \quad \text{and} \quad m_{\mathfrak{t}^\nu} L'_k = q^{\mathrm{c}_{\mathfrak{t}^\nu}(k)} m_{\mathfrak{t}^\nu}.$$

Proof. The first identity follows from [**17**, Theorem 3.32]. The second identity follows from the first using the fact that $L'_k = (q-1)L_k + 1$. □

Lemma 3.3. *Suppose that* $1 \le i \le i' \le n+\gamma-1$ *and* $1 \le j, j' \le n+\gamma$*. Then*

a) $L_j L_{j'} = L_{j'} L_j$,
b) $T_i L_j = L_j T_i$ *if* $i \ne j, j-1$,
c) $T_i L_i = L_{i+1} T_i - L'_{i+1}$,
d) $T_i L_{i+1} = L'_{i+1} + L_i T_i$,
e) $T_i(L_i + L_{i+1}) = (L_i + L_{i+1})T_i$,
f) $T_i L_i L_{i+1} = L_i L_{i+1} T_i$,
g) $T_{i,i'} L_{i'} = L_i T_{i,i'} + \sum_{x=i+1}^{i'} L'_x T_{i,i'\setminus x}$.

Proof. All but the last identity are given in [**17**, Proposition 3.26 and Exercise 3.6]. Part (g) is readily proved by induction on $i'-i$. □

Suppose that α is a partition and that β is a composition of an integer m and let S be an α-tableau of type β. Recall from Section 2.4 that

$$m_{\mathsf{S}} = \sum_{\substack{\mathfrak{s} \in \mathrm{RStd}(\alpha)\\ \beta(\mathfrak{s}) = \mathsf{S}}} m_{\mathfrak{s}}.$$

By definition $m_{\mathsf{S}} \in S^\alpha$. We need a different description of m_{S}.

Define $\dot{\mathsf{S}}$ to be the unique row standard tableau such that $\beta(\dot{\mathsf{S}}) = \mathsf{S}$ and the numbers in each row of $\mathfrak{t}^\beta$ appear in row order in $\dot{\mathsf{S}}$. Then $d(\dot{\mathsf{S}})$ is the unique element of minimal length in the double coset $\mathfrak{S}_\alpha d(\dot{\mathsf{S}}) \mathfrak{S}_\beta$ by [**17**, Prop. 4.4], and by [**17**, (4.6)]

$$m_{\mathsf{S}} = m_{\dot{\mathsf{S}}} \sum_{w \in \mathcal{D}_{\mathsf{S}}} T_w,$$

where $\mathcal{D}_{\mathsf{S}}$ is the set of all $w \in \mathfrak{S}_\beta$ such that if $i < j$ lie in the same row of $\dot{\mathsf{S}}w$ then $(i)w < (j)w$. In fact, by [**17**, Prop. 4.4] again, $\mathcal{D}_{\mathsf{S}} = \mathcal{D}_\sigma \cap \mathfrak{S}_\beta$ where the composition σ is given by $\mathfrak{S}_\sigma = d(\dot{\mathsf{S}})^{-1} \mathfrak{S}_\alpha d(\dot{\mathsf{S}}) \cap \mathfrak{S}_\beta$ and $\mathcal{D}_\sigma = \{\, d(\mathfrak{s}) \mid \mathfrak{s} \in \mathrm{RStd}(\sigma)\,\}$ is the set of distinguished (or minimal length) right coset representatives of $\mathfrak{S}_\sigma$ in $\mathfrak{S}_n$. Write $\beta = (\beta_1, \dots, \beta_b)$. Then $\mathfrak{S}_\beta = \mathfrak{S}_{\beta_1} \times \dots \times \mathfrak{S}_{\beta_b}$ and every element w of $\mathfrak{S}_\beta$ can be written uniquely as a product of commuting permutations $w = w_1 \dots w_b$ where, abusing notation slightly, $w_i \in \mathfrak{S}_{\beta_i}$ for $1 \le i \le b$. Let $\mathcal{D}_{\mathsf{S}}(i) = \mathcal{D}_{\mathsf{S}} \cap \mathfrak{S}_{\beta_i}$ for $1 \le i \le b$. Define $D_{\mathsf{S}} = D_{\mathsf{S}}(1) \dots D_{\mathsf{S}}(b)$, where $D_{\mathsf{S}}(i) = \sum_{w \in \mathcal{D}_{\mathsf{S}}(i)} T_w$. Then we have

$$m_{\mathsf{S}} = m_{\dot{\mathsf{S}}} D_{\mathsf{S}} = m_{\mathfrak{t}^\alpha} T_{d(\dot{\mathsf{S}})} D_{\mathsf{S}}. \tag{3.4}$$

3.5. **Example** Suppose that $\alpha=(7,2)$, $\beta=(4,3,2)$ and $\mathsf{S}=\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3\\\hline 1&2\\\cline{1-2}\end{array}\in\operatorname{RStd}(\alpha,\beta)$. Then $\dot{\mathsf{S}}=\begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&5&6&8&9\\\hline 4&7\\\cline{1-2}\end{array}$ and

$$m_{\mathsf{S}}=m_{\mathfrak{t}^\alpha}T_{7,8}T_{6,7}T_5T_4(1+T_3+T_3T_2+T_3T_2T_1)(1+T_6+T_6T_5).$$

$\Diamond$

Lemma 3.6. *Let a,b,c and g are integers with $1\le a<c<b\le m$ and $g\notin\{a,\dots,b\}$ and let $\beta=(1^{a-1},b-a+1,1^{m-b})$, a composition of m. Suppose α is a partition of m and that $\mathfrak{t}$ is a row-standard α-tableau such that $a,\dots,b$ are in row order in $\mathfrak{t}$, $\operatorname{row}_{\mathfrak{t}}(c-1)<\operatorname{row}_{\mathfrak{t}}(c)$ and $i'=\operatorname{row}_{\mathfrak{t}}(g)<i=\operatorname{row}_{\mathfrak{t}}(c)$. Let $\mathfrak{s}=\mathfrak{t}(c,g)$, $\mathsf{T}=\beta(\mathfrak{t})$ and $\mathsf{S}=\beta(\mathfrak{s})$. Then*

$$m_{\mathfrak{s}}\Big(\sum_{j=c}^{c+l}T_{c,j}\Big)D_{\mathsf{T}}(a)=q^s[\mathsf{S}^a_{i'}]\,m_{\mathsf{S}},$$

where $l=\mathsf{S}^a_i$ and $s=\mathsf{S}^a_{(i',i)}$.

Proof. We prove the Lemma using some standard properties of the distinguished coset representatives of Coxeter groups. To exploit these results it is convenient to introduce some new notation.

If σ is a composition of m let $J_\sigma=\{\,1\le i<m\mid \operatorname{row}_{\mathfrak{t}^\sigma}(i)=\operatorname{row}_{\mathfrak{t}^\sigma}(i+1)\,\}$. Then $\mathfrak{S}_\sigma$ is generated by $\{\,(i,i+1)\mid i\in J_\sigma\,\}$ and the map $\sigma\mapsto J_\sigma$ defines a bijection between the set of compositions of m and the subsets of $\Pi_m=\{1,2,\dots,m-1\}$. If $J=J_\sigma\subseteq\Pi_m$ set $m_J=m_\sigma$, $\mathfrak{S}_J=\mathfrak{S}_\sigma$, $\mathcal{D}_J=\mathcal{D}_\sigma$ and $D_J=D_\sigma$. If $J\subseteq K\subseteq\Pi_m$ set $\mathcal{D}_J^K=\mathcal{D}_J\cap\mathfrak{S}_K$. Then $\mathcal{D}_J^K$ is a complete set of coset representatives for $\mathfrak{S}_J$ in $\mathfrak{S}_K$ and, moreover, the following two properties hold:

(D1) Suppose that $J\subseteq K\subseteq A\subseteq\Pi_m$. Then $D_J^A=D_J^KD_K^A$.
(D2) Suppose that $J,K,L\subseteq\Pi_m$ with $J\subseteq K$ and $|k-l|>1$ for all $k\in K$ and $l\in L$. Then $D_J^K=D_{J\cup L}^{K\cup L}$.

Property (D1) is well-known and easy to prove: see, for example, [**1**, Lemma 2.1]. The second statement (D2) is trivial because the assumptions imply that $\mathfrak{S}_{K\cup L}=\mathfrak{S}_K\times\mathfrak{S}_L$ and $\mathfrak{S}_{J\cup L}=\mathfrak{S}_J\times\mathfrak{S}_L$.

Let $A=\{a,a+1,\dots,b-1\}$ and let $E=\{\,e\in A\mid\operatorname{row}_{\mathfrak{t}}(e)=\operatorname{row}_{\mathfrak{t}}(e+1)\,\}$. Then $\mathcal{D}_{\mathsf{T}}(a)=\mathcal{D}_{\mathsf{T}}=\mathcal{D}_E^A$. Similarly, let

$$\begin{aligned}E'&=\{\,e\in A\mid\operatorname{row}_{\mathsf{S}}(e)=\operatorname{row}_{\mathsf{S}}(e+1)\,\}\\&=\{\,e\in E\mid\operatorname{row}_{\mathfrak{t}}(e)\notin(i',i)\,\}\cup\{\,e+1\mid e\in E\text{ and }\operatorname{row}_{\mathfrak{t}}(e)\in[i',i)\,\}\setminus\{c\},\end{aligned}$$

Then $\mathcal{D}_{\mathsf{S}}(a)=\mathcal{D}_S=\mathcal{D}_{E'}^A$. To prove the Lemma we consider various subsets of A which depend on E and E'. Let

$$C=\{\,e\in E\cap E'\mid\operatorname{row}_{\mathfrak{t}}(e)=i\,\}\quad\text{and}\quad C'=\{\,e\in E\cap E'\mid\operatorname{row}_{\mathfrak{t}}(e)=i'\,\}$$

and let $L,L'\subseteq A$ be the subsets of A such that

$$E=C\sqcup\{c\}\sqcup L\qquad\text{and}\qquad E'=C'\sqcup\{c'\}\sqcup L',\qquad\text{(disjoint unions)},$$

where $c'\in A$ is maximal such that $\operatorname{row}_{\mathfrak{t}}(c')=i'$. Note that $c'\le c$ and $\mathsf{S}^a_{(i',i)}=c-c'$. In particular, $c=c'$ if and only if $s=\mathsf{S}^a_{(i',i)}=0$.

Armed with these definitions we can now prove the lemma. We have

$$\begin{aligned}m_{\mathfrak{s}}\Big(\sum_{j=c}^{c+l}T_{c,j}\Big)D_{\mathsf{T}}(a)&=m_{\dot{\mathsf{S}}}T_{c',c}D_C^{C\cup\{c\}}D_E^A=m_{\dot{\mathsf{S}}}T_{c',c}D_{C\cup L}^ED_E^A\\&=m_{\dot{\mathsf{S}}}T_{c',c}D_{C\cup L}^A\end{aligned}$$

where the last two equalities follow by (D2) and (D1), respectively. Let $d = (c', c'+1)\dots(c-1,c)$ so that $T_{c',c} = T_d$. Then $\mathfrak{S}_{C\cup L} = d^{-1}\mathfrak{S}_{C'\cup L'}d$ and $d\in\mathcal{D}_{C'\cup L'}\cap\mathcal{D}_{C\cup L}^{-1}$ so that $m_{C'\cup L'}T_d = T_d m_{C\cup L}$. (In fact, $\mathcal{D}^A_{C'\cup L'} = d\mathcal{D}^A_{C\cup L}$ by [**1**, Lemma 2.4], however, this is not enough for our purposes because , in general, $D^A_{C'\cup L'}\neq T_d D^A_{C\cup L}$.) Now, $m_{\dot{\mathsf{S}}}T_w = q^{\ell(w)}m_{\dot{\mathsf{S}}}$ for all $w\in\mathfrak{S}_{C'\cup L'}$, so $m_{\dot{\mathsf{S}}} = hm_{C'\cup L'}$ for some $h\in\mathscr{H}_m^{\mathcal{Z}}$. Consequently, continuing the last displayed equation,

$$\begin{aligned} m_{\mathfrak{s}}\Big(\sum_{j=c}^{c+l}T_{c,j}\Big)D_{\mathsf{T}}(a) &= hm_{C'\cup L'}T_dD^A_{C\cup L} = hT_dm_{C\cup L}D^A_{C\cup L}\\ &= hT_dm_A = q^{\ell(d)}hm_A = q^{\ell(d)}hm_{C'\cup L'}D^A_{C'\cup L'}. \end{aligned}$$

Observe that $\ell(d) = c-c' = \mathsf{S}^a_{(i',i)} = s$. Therefore, using (D1) and (D2) again,

$$\begin{aligned} m_{\mathfrak{s}}\Big(\sum_{j=c}^{c+l}T_{c,j}\Big)D_{\mathsf{T}}(a) &= q^s m_{\dot{\mathsf{S}}}D^{E'}_{C'\cup L'}D^A_{E'} = q^s m_{\dot{\mathsf{S}}}D_{C'}^{C'\cup\{c'\}}D^A_{E'}\\ &= q^s[\mathsf{S}^a_{i'}]m_{\dot{\mathsf{S}}}D^A_{E'} \end{aligned}$$

where the last equality follows because $m_{\dot{\mathsf{S}}}T_w = q^{\ell(w)}m_{\dot{\mathsf{S}}}$ for all $w\in\mathfrak{S}_{C'\cup\{c'\}}$ by Lemma 3.1. We have already observed that $\mathcal{D}_{\mathsf{S}} = \mathcal{D}^A_{E'}$, so an application of (3.4) now completes the proof. □

3.7. **Example** Suppose that $a=4$, $b=9$, $c=8$ and that $g=3$. Then

$$\mathfrak{t} = \begin{array}{llll}1&3&4&5\\2&6&7\\8&9\end{array} \quad\Longrightarrow\quad \mathfrak{s} = \begin{array}{llll}1&4&5&8\\2&6&7\\3&9\end{array},\quad \mathsf{T} = \begin{array}{llll}1&3&4&4\\2&4&4\\4&4\end{array}\quad\text{and}\quad \mathsf{S} = \begin{array}{llll}1&4&4&4\\2&4&4\\3&4\end{array}.$$

Abusing notation and identifying $m_{\mathfrak{s}}$ with $\mathfrak{s}$ and S with m_{S}, we have

$$\begin{array}{llll}1&4&5&8\\2&6&7\\3&9\end{array}\ (1+T_8)D_{\mathsf{T}}(4) = q^2[3]\begin{array}{llll}1&4&4&4\\2&4&4\\3&4\end{array}$$

where $D_{\mathsf{T}}(4) = \sum_{w\in\mathcal{D}_{\mathsf{T}}}T_w$. By definition, $\mathcal{D}_{\mathsf{T}}(4) = \mathcal{D}_{\mathsf{T}}$ is the set of minimal length coset representatives of $\mathfrak{S}_{\{4,5\}}\times\mathfrak{S}_{\{6,7\}}\times\mathfrak{S}_{\{8,9\}}$ in $\mathfrak{S}_{\{4,\dots,9\}}$. ◊

For any composition $\sigma = (\sigma_1,\sigma_2,\dots)$ let $\overline{\sigma}_k = \sigma_1+\dots+\sigma_k$, for $k\geq 0$.

Lemma 3.8. *Suppose that $\eta\subseteq\nu$ is a partition of n and set $\xi = (\nu_1-\eta_1,\nu_2-\eta_2,\dots)$, a composition of γ. Then*

$$m_{\mathfrak{t}^\nu_\eta} = m_{\mathfrak{t}^\nu}\prod_{i=0}^{z-1}\prod_{k=0}^{\xi_i-1}T_{\overline{\nu}_{z-i}-k,\,n+\overline{\xi}_{z-i}-k}.$$

Proof. For $0\leq j\leq\gamma$, let $\mathfrak{t}(j)$ be the ν-tableau such that the entries $n+j+1,\dots,n+\gamma$ appear in the same position that they appear in $\mathfrak{t}^\nu_\eta$ and the entries $1,2,\dots,n+j$ are in row order. Consider $\mathfrak{t}(\gamma-1)$. Suppose that $n+\gamma$ appears (at the end of) row r in $\mathfrak{t}^\nu_\eta$. Then $m_{\mathfrak{t}(\gamma-1)} = m_{\mathfrak{t}^\nu}T_{\overline{\nu}_r}\dots T_{n+\gamma-1} = m_{\mathfrak{t}^\nu}T_{\overline{\nu}_r,n+\gamma}$ by Lemma 3.1. The general case now follows by downwards induction on j using essentially the same observations. □

Similarly, it is straightforward to check the following lemma.

Lemma 3.9. *Suppose $\mathfrak{t}\in\operatorname{RStd}(\nu)$ and let $\eta = \operatorname{Shape}(\mathfrak{t}_{\downarrow n})$. Then*

$$m_{\mathfrak{t}} = m_{\mathfrak{t}^\nu}T_{d(\mathfrak{t}^\nu_\eta)}T_w$$

for a unique permutation $w\in\mathfrak{S}_n\times\mathfrak{S}_\gamma$.

We are now ready to start proving the main results of this section. Recall that if $\eta\subseteq\nu$ is a partition of n then the almost initial tableau $\mathfrak{t}^\nu_\eta$ was defined in Section 2.4. If $1\leq r\leq z$ then define $c^\eta_r = \eta_r - r$.

Lemma 3.10. *Suppose that* $\mathfrak{t} = \mathfrak{t}^{\nu}_{\eta}$ *is an almost initial tableau such that* $\mathrm{row}_{\mathfrak{t}}(n+1) \neq z$ *and let* $j \geq 1$ *be maximal such that* $r = \mathrm{row}_{\mathfrak{t}}(n+j) < z$. *For* $i \geq 1$ *set* $\xi_i = \nu_i - \eta_i$ *and if* $1 \leq g \leq n$ *then let* $c(g) = c^{\eta}_{m}$ *where* $\mathrm{row}_{\mathfrak{t}}(g) = m$. *Then*

$$m_{\mathfrak{t}} L_{n+j} = [\mathrm{c}_{\mathfrak{t}}(n+j)] m_{\mathfrak{t}} + q^{\xi_r - 1} \sum_{g=\overline{\nu}_r - j+1}^{n} q^{c(g)} m_{\mathfrak{t}(g,n+j)}.$$

Proof. Using in turn, Lemma 3.8, Lemma 3.3(g) and Lemma 3.2, we find

$$\begin{aligned}
m_{\mathfrak{t}} L_{n+j} &= \Big(m_{\mathfrak{t}^{\nu}} \prod_{i=0}^{r-1} \prod_{k=0}^{\xi_{r-i}-1} T_{\overline{\nu}_{r-i}-k, n+\overline{\xi}_{r-i}-k} \Big) L_{n+j} \\
&= m_{\mathfrak{t}^{\nu}} T_{\overline{\nu}_r, n+j} L_{n+j} \Big(\prod_{k=1}^{\xi_r - 1} T_{\overline{\nu}_r - k, n+j-k} \Big) \Big(\prod_{i=1}^{r-1} \prod_{k=0}^{\xi_{r-i}-1} T_{\overline{\nu}_{r-i}-k, n+\overline{\xi}_{r-i}-k} \Big) \\
&= m_{\mathfrak{t}^{\nu}} \Big(L_{\overline{\nu}_r} T_{\overline{\nu}_r, n+j} + \sum_{x=\overline{\nu}_r+1}^{n+j} L'_x T_{\overline{\nu}_r, n+j \setminus x} \Big) \Big(\prod_{k=1}^{\xi_r - 1} T_{\overline{\nu}_r - k, n+j-k} \Big) \\
&\qquad \times \Big(\prod_{i=1}^{r-1} \prod_{k=0}^{\xi_{r-i}-1} T_{\overline{\nu}_{r-i}-k, n+\overline{\xi}_{r-i}-k} \Big) \\
&= [\mathrm{c}_{\mathfrak{t}}(n+j)] m_{\mathfrak{t}} + m_{\mathfrak{t}^{\nu}} \sum_{x=\overline{\nu}_r+1}^{n+j} q^{\mathrm{c}_{\mathfrak{t}^{\nu}}(x)} T_{\overline{\nu}_r, n+j \setminus x} \Big(\prod_{k=1}^{\xi_r - 1} T_{\overline{\nu}_r - k, n+j-k} \Big) \\
&\qquad \times \Big(\prod_{i=1}^{r-1} \prod_{k=0}^{\xi_{r-i}-1} T_{\overline{\nu}_{r-i}-k, n+\overline{\xi}_{r-i}-k} \Big).
\end{aligned}$$

Now fix x with $\overline{\nu}_r + 1 \leq x \leq n+j$. To complete the proof, we show that

$$\begin{aligned}
q^{\mathrm{c}_{\mathfrak{t}^{\nu}}(x)} m_{\mathfrak{t}^{\nu}} T_{\overline{\nu}_r, n+j \setminus x} \Big(\prod_{k=1}^{\xi_r - 1} T_{\overline{\nu}_r - k, n+j-k} \Big) \Big(\prod_{i=1}^{r-1} \prod_{k=0}^{\xi_{r-i}-1} T_{\overline{\nu}_{r-i}-k, n+\overline{\xi}_{r-i}-k} \Big) \\
= q^{\xi_r - 1} q^{c(x-j)} m_{\mathfrak{t}(x-j, n+j)}.
\end{aligned}$$

Note that x lies in the same position of $\mathfrak{t}^{\nu}$ that $x-j$ lies in $\mathfrak{t}$. Let $\mathrm{row}_{\mathfrak{t}^{\nu}}(x) = m$. Therefore

$$\begin{aligned}
m_{\mathfrak{t}^{\nu}} T_{\overline{\nu}_r, n+j \setminus x} &= m_{\mathfrak{t}^{\nu}(x-1, x-2, \dots, \overline{\nu}_r)} T_{x, n+j} \\
&= q^{c(x-j) - \mathrm{c}_{\mathfrak{t}^{\nu}}(x)} m_{\mathfrak{t}^{\nu}(x-1, x-2, \dots, \overline{\nu}_r)} T_{\overline{\nu}_m, n+j} \\
&= q^{c(x-j) - \mathrm{c}_{\mathfrak{t}^{\nu}}(x)} m_{\mathfrak{t}'}
\end{aligned}$$

where $\mathfrak{t}' = \mathfrak{t}^{\nu}(x-1, x-2, \dots, \overline{\nu}_r)(n+j, n+j-1, \dots, \overline{\nu}_m)$. Using induction on ε, where $1 \leq \varepsilon \leq \xi_r$, it follows that

$$m_{\mathfrak{t}'} \Big(\prod_{k=1}^{\varepsilon - 1} T_{\overline{\nu}_r - k, n+j-k} \Big) = m_{\mathfrak{t}'} \Big(\prod_{k=1}^{\varepsilon - 1} q T_{\overline{\nu}_r - k, n+j-k \setminus x-k} \Big).$$

Applying a second inductive argument, we find

$$m_{\mathfrak{t}'} \Big(\prod_{k=1}^{\xi_r - 1} T_{\overline{\nu}_r - k, n+j-k \setminus x-k} \Big) \Big(\prod_{i=1}^{r-1} \prod_{k=0}^{\xi_{r-i}-1} T_{\overline{\nu}_{r-i}-k, n+\overline{\xi}_{r-i}-k} \Big) = m_{\mathfrak{t}(x-j, n+j)}.$$

The result follows. □

Suppose $1 \leq u \leq v \leq n$ and that $\pi \in \mathfrak{S}_n$. Let $\mathcal{D}(u, v, \pi)$ be the set of tuples $\mathbf{p} = (p_0, p_1, \dots, p_\epsilon)$ such that $u - 1 = p_0 < p_1 < p_2 < \dots < p_\epsilon = v$ and $(p_1)\pi >$

$(p_2)\pi > \ldots > (p_\epsilon)\pi$. For each $\mathbf{p} \in \mathcal{D}(u,v,\pi)$ let $\check{\mathbf{p}}$ be the permutation $(p_1, p_1-1, \ldots, p_0+1)(p_2, p_2-1, \ldots, p_1+1)\ldots(p_\epsilon, p_\epsilon-1, \ldots, p_{\epsilon-1}+1)$. Let $\ell(\mathbf{p}) = \epsilon - 1$ and

$$b(\mathbf{p}) = \sum_{i=0}^{\epsilon-1} \#\{\, j \mid p_i < j < p_{i+1} \text{ and } (j)\pi > (p_{i+1})\pi \,\}.$$

Lemma 3.11. *Suppose* $1 \le u \le v \le n$ *and that* $\pi \in \mathfrak{S}_n$. *Then*

$$T_{u,v}T_\pi = \sum_{\mathbf{p}\in\mathcal{D}(u,v,\pi)} q^{b(\mathbf{p})}(q-1)^{\ell(\mathbf{p})} T_{\check{\mathbf{p}}\pi}.$$

Proof. We use induction on $v-u$, the case $u=v$ being trivial. Assume $v-u \ge 1$ and that the lemma holds for $v-u-1$. By induction,

$$T_{u,v}T_\pi = \sum_{\mathbf{p}\in\mathcal{D}(u+1,v,\pi)} q^{b(\mathbf{p})}(q-1)^{\ell(\mathbf{p})} T_u T_{\check{\mathbf{p}}\pi}.$$

If $\mathbf{p} = (p_0, p_1, \ldots, p_\epsilon) \in \mathcal{D}(u+1, v, \pi)$ then

$$T_u T_{\check{\mathbf{p}}\pi} = \begin{cases} T_{\check{\mathbf{p}}'\pi}, & (u)\pi < (p_1)\pi, \\ qT_{\check{\mathbf{p}}'\pi} + (q-1)T_{\check{\mathbf{p}}''\pi}, & (u)\pi > (p_1)\pi, \end{cases}$$

where $\mathbf{p}' = (u-1, p_1, \ldots, p_\epsilon)$ and $\mathbf{p}'' = (u-1, p_0, p_1, \ldots, p_\epsilon)$. The result follows. $\square$

3.2. **Bumping tableaux.** In this section we prove a series of ‘bumping lemmas’ which culminate in the proof of Proposition 3.19. This result contains Proposition 2.5 as a special case, so it completes the proof of Theorem 2.7. Throughout this section, ν is an arbitrary partition of $n+\gamma$.

Suppose that $\mathfrak{t} \in \operatorname{RStd}(\nu)$. Suppose that $1 \le j \le n+\gamma$ and that $\operatorname{row}_\mathfrak{t}(j) = r$. Say that $\mathfrak{s}$ is obtained from $\mathfrak{t}$ by **bumping** j down $\mathfrak{t}$ if there exists $\epsilon \ge 1$ and integers $r = r_0 < r_1 < \ldots < r_\epsilon \le z$ and $j > d_1 > \ldots > d_\epsilon \ge 1$ such that $\operatorname{row}_\mathfrak{t}(d_i) = r_i$ for $1 \le i \le \epsilon$ and $\mathfrak{s} = \mathfrak{t}(j, d_1, \ldots, d_\epsilon)$. If $\mathfrak{s}$ is such a tableau, write $\mathfrak{s} \prec_j \mathfrak{t}$. Define $\ell_\mathfrak{t}(\mathfrak{s}) = \epsilon - 1$ and

$$\begin{aligned} b_\mathfrak{t}^\mathfrak{s} &= c^\eta_{r_\epsilon} - \epsilon + \sum_{i=0}^{\epsilon-1} \#\{\, j \mid r_i \le \operatorname{row}_\mathfrak{t}(j) < r_{i+1} \text{ and } j > d_{i+1} \,\} \\ &= c^\eta_{r_\epsilon} - \epsilon + \sum_{i=0}^{\epsilon-1} \mathfrak{s}^{>d_{i+1}}_{[r_i, r_{i+1})}. \end{aligned}$$

The notation $\mathfrak{s}^{>d_{i+1}}_{[r_i, r_{i+1})}$ was introduced in Section 2.2.

Lemma 3.12. *Suppose* $\mathfrak{t} \in \operatorname{RStd}(\nu)$ *is such that* $\eta = \operatorname{Shape}(\mathfrak{t}_{\downarrow n}) \ne \mu$ *and the entries* $n+1, n+2, \ldots, n+\gamma$ *are in row order. Choose* j *maximal such that* $r = \operatorname{row}_\mathfrak{t}(n+j) < z$. *Then*

$$m_\mathfrak{t}(L_{n+j} - [c_r]) = \sum_{\mathfrak{s}\prec_{n+j}\mathfrak{t}} q^{b_\mathfrak{t}^\mathfrak{s}}(q-1)^{\ell_\mathfrak{t}(\mathfrak{s})} m_\mathfrak{s}.$$

Proof. Following Lemma 3.9, let π be the permutation such that $m_\mathfrak{t} = m_{\mathfrak{t}^\nu_\eta} T_\pi$. Since $\pi \in \mathfrak{S}_n$ we have that $m_\mathfrak{t} L_{n+j} = m_{\mathfrak{t}^\nu_\eta} L_{n+j} T_\pi$. We apply Lemma 3.10, keeping the notation of that lemma, except that we set $V = \overline{\nu}_r - j + 1$. For $V \le g \le n$, let $\sigma_g =$

$\mathrm{Shape}(\mathfrak{t}(g,n+j)_{\downarrow n})$. Then

$$\begin{aligned} m_{\mathfrak{t}}(L_{n+j}-[c_r]) &= m_{\mathfrak{t}^\nu_\eta}(L_{n+j}-[c_r])T_\pi \\ &= q^{\xi_r-1}\sum_{g=V}^{n} q^{c(g)} m_{\mathfrak{t}(g,n+j)}T_\pi \\ &= q^{\xi_r-1}\sum_{g=V}^{n} q^{c(g)} m_{\mathfrak{t}^\nu_{\sigma_g}}T_{V,g}T_\pi \\ &= q^{\xi_r-1}\sum_{g=V}^{n} q^{c(g)} \sum_{\mathbf{p}\in\mathcal{D}(V,g,\pi)} q^{b(\mathbf{p})}(q-1)^{\ell(\mathbf{p})} m_{\mathfrak{t}^\nu_{\sigma_g}}T_{\check{\mathbf{p}}\pi} \end{aligned}$$

by Lemma 3.11. Now notice that there is a bijection

$$\{\,\mathfrak{s}\mid\mathfrak{s}\prec_{n+j}\mathfrak{t}\,\}\overset{\sim}{\longleftrightarrow}\{\,(g,\mathbf{p})\mid V\le g\le n \text{ and } \mathbf{p}\in\mathcal{D}(V,g,\pi)\,\}$$

given as follows. For each pair $(g,\mathbf{p})$ as above, let $\mathbf{d}=(d_1,\dots,d_\epsilon)$ where $d_i=(p_i)\pi$ for $1\le i<\epsilon$ and $d_\epsilon=(g)\pi$. By construction, $n+j>d_1>\dots>d_\epsilon$ and if $1\le i<j\le\epsilon$ then $(p_i)\pi>(p_j)\pi$ and so $\mathrm{row}_{\mathfrak{t}}(i)>\mathrm{row}_{\mathfrak{t}}(j)$. Thus $\mathfrak{s}=\mathfrak{t}(n+j,d_1,\dots,d_\epsilon)$ is formed by bumping $n+j$ down $\mathfrak{t}$. Under this correspondence, since $\check{\mathbf{p}}\pi\in\mathfrak{S}_n$, in order to see that

$$m_{\mathfrak{t}^\nu_{\sigma_g}}T_{\check{\mathbf{p}}\pi}=m_{\mathfrak{t}(n+j,d_1,\dots,d_\epsilon)}$$

it is enough to observe that the permutations $d(\mathfrak{t}^\nu_{\sigma_g})\check{\mathbf{p}}\pi$ and $(n+j,d_1,\dots,d_\epsilon)$ agree. It remains to check that

$$q^{\xi_r-1}q^{c(g)}q^{b(\mathbf{p})}(q-1)^{\ell(\mathbf{p})}=q^{b^{\mathfrak{s}}_{\mathfrak{t}}}(q-1)^{\ell_{\mathfrak{t}}(s)},$$

which again follows from the definitions. □

Now suppose that T is a ν-tableau of arbitrary type which contains an entry equal to k in row r. We generalize the notion of bumping by saying that a tableau U is obtained from T by **bumping** k **from row** r if there exist an integer $\epsilon\ge1$ and integers $r=r_0<r_1<\dots<r_\epsilon\le z$ and $k>d_1>\dots>d_\epsilon$ such that for $1\le i\le\epsilon$, row r_i of T contains an entry equal to d_i and U is obtained by repeatedly exchanging k in row r_i with d_{i+1} in row r_{i+1}. If U is obtained from T in this way, write $\mathsf{U}\prec_{k,r}\mathsf{T}$. We suppress r if T contains only one entry equal to k. Define $\ell_{\mathsf{T}}(\mathsf{U})=\epsilon-1$, $f^{\mathsf{U}}_{\mathsf{T}}=\prod_{i=0}^{\ell_{\mathsf{T}}(\mathsf{U})}[\mathsf{U}^{d_{i+1}}_{r_i}]$ and

$$b^{\mathsf{U}}_{\mathsf{T}}=c^\eta_{r_\epsilon}+\sum_{i=0}^{\epsilon-1}\Big(\mathsf{U}^{>d_{i+1}}_{r_i}+\mathsf{U}^{\ge d_{i+1}}_{(r_i,r_{i+1})}\Big).$$

This agrees with the previous definition of $b^{\mathsf{U}}_{\mathsf{T}}$ when T is a tableau of type $(1^{n+\gamma})$.

Define a ν-tableau T to be **basic** if it is a semistandard tableau of type $\eta+1^\gamma$ for some partition η of n such that $\eta\subseteq\nu$ and the entries $z+1,z+2,\dots,z+\gamma$ are in row order. Note that for $1\le j\le\gamma$, the position of $z+j$ in T is the same as the position of $n+j$ in $\dot{\mathsf{T}}$.

Corollary 3.13. *Suppose that* T *is a basic tableau of type* $\eta+1^\gamma$ *such that* $\eta\ne\mu$. *Let* j *be maximal such that* $r=\mathrm{row}_{\mathsf{T}}(z+j)<z$. *Then*

$$m_{\mathsf{T}}\big(L_{n+j}-[c_r]\big)=\sum_{\mathsf{U}\prec_{n+j}\mathsf{T}}q^{b^{\mathsf{U}}_{\mathsf{T}}}(q-1)^{\ell_{\mathsf{T}}(\mathsf{U})}f^{\mathsf{U}}_{\mathsf{T}}m_{\mathsf{U}}.$$

Proof. Let $\mathfrak{t}=\dot{\mathsf{T}}=\mathfrak{t}^\nu_\eta$, so that $m_{\mathsf{T}}=m_{\mathfrak{t}}D_{\mathsf{T}}$ by (3.4). Keeping the notation of Lemma 3.12 we have

$$\begin{aligned} m_{\mathsf{T}}(L_{n+j}-[c_r]) &= m_{\mathfrak{t}}(L_{n+j}-[c_r])D_{\mathsf{T}} \\ &= \sum_{\mathfrak{s}\prec_{n+j}\mathfrak{t}} q^{b^{\mathfrak{s}}_{\mathfrak{t}}}(q-1)^{\ell_{\mathfrak{t}}(\mathfrak{s})}m_{\mathfrak{s}}D_{\mathsf{T}}. \end{aligned}$$

Now apply Lemma 3.6 and the definitions. □

Lemma 3.14. *Suppose that* T *is a basic tableau of type* $\eta+1^\gamma$ *such that* $z+j$ *lies in row* z *and that* $c\in\mathbb{Z}$. *Then* $m_{\mathsf{T}}(L_{n+j}-[c]) = q^c[c_z-c-\gamma+j]m_{\mathsf{T}}$.

Proof. It follows from Lemma 3.2 and the proof of Lemma 3.12 that $m_{\mathsf{T}}(L_{n+j}-[c]) = ([c_z-\gamma+j]-[c])m_{\mathsf{T}} = q^c[c_z-c-\gamma+j]m_{\mathsf{T}}$. □

Before generalizing the previous results to bumping tableaux we take a break and prove the following useful Gaussian integer identity.

Lemma 3.15. *Suppose that* $v\ge r\ge 0$ *and that* $C_x, U_x\in\mathbb{Z}$, *for* $1\le x\le v$. *Then*

$$\sum_{x=r+1}^{v}\left(\prod_{y=1}^{x-1} q^{U_y}[C_y]\right)[U_x]\left(\prod_{y=x+1}^{v}[C_y+U_y]\right)+\prod_{y=r+1}^{v} q^{U_y}[C_y] = \prod_{y=r+1}^{v}[C_y+U_y].$$

Proof. The integer r plays no essential role so we can, and do, assume that $r=0$. We claim that for $1\le m\le v$ we have

$$\begin{aligned}\sum_{x=m}^{v}\left(\prod_{y=1}^{x-1} q^{U_y}[C_y]\right)[U_x]\left(\prod_{y=x+1}^{v}[C_y+U_y]\right)&+\prod_{y=1}^{v} q^{U_y}[C_y]\\ &= \prod_{y=1}^{m-1} q^{U_y}[C_y]\cdot\prod_{y=m}^{v}[C_y+U_y].\end{aligned}$$

The lemma follows directly from the claim. To prove the claim, we use downwards induction on m. If $m=v$ then the equation gives

$$\left(\prod_{y=1}^{v-1} q^{U_y}[C_y]\right)[U_v]+\prod_{y=1}^{v} q^{U_y}[C_y] = \left(\prod_{y=1}^{v-1} q^{U_y}[C_y]\right)[C_y+U_y].$$

Now suppose $1\le m<v$ and the claim holds for $m+1$. Then

$$\begin{aligned}&\sum_{x=m}^{v}\left(\prod_{y=1}^{x-1} q^{U_y}[C_y]\right)[U_x]\left(\prod_{y=x+1}^{v}[C_y+U_y]\right)+\prod_{y=1}^{v} q^{U_y}[C_y]\\ &\quad=\left(\prod_{y=1}^{m-1} q^{U_y}[C_y]\right)[U_m]\left(\prod_{y=m+1}^{v}[C_y+U_y]\right)+\prod_{y=1}^{m} q^{U_y}[C_y]\cdot\prod_{y=m+1}^{v}[C_y+U_y]\\ &\quad=\prod_{y=1}^{m-1} q^{U_y}[C_y]\cdot\prod_{y=m}^{v}[C_y+U_y].\end{aligned}$$

This completes the proof of the claim and hence the lemma. □

Suppose that T is a ν-tableau of arbitrary type which contains an entry equal to k in row r. We say that a tableau U is obtained by **weakly bumping** k **from row** r **into row** z if there exist an integer $\epsilon\ge 1$ and integers $r=r_0<r_1<\dots<r_\epsilon=z$ and $d_1,d_2,\dots,d_\epsilon$ such that for $1\le i\le\epsilon$, we have $k>d_i$ and row r_i of T contains an entry equal to d_i, and U is obtained by repeatedly exchanging k in row r_i with d_{i+1} in row r_{i+1}. We write $\mathsf{U}\prec^{\mathrm{w}}_{k,r}\mathsf{T}$. Once again, we suppress r if T contains only one entry equal to k.

Remark. The differences between bumping k from row r and weakly bumping k from row r into row z are that, when $\mathsf{U}\prec^{\mathrm{w}}_{k,r}\mathsf{T}$, we do not insist that $d_1>d_2>\dots>d_\epsilon$ but we do insist that $r_\epsilon=z$.

If $\mathsf{U}\prec^{\mathrm{w}}_{k,r}\mathsf{T}$ then the integers d_i, r_i above are not necessarily unique. Nonetheless, there is a unique sequence $\mathbf{a}^{\mathsf{U}}_{\mathsf{T}}=(a_{r+1},\dots,a_z)$; namely, if $r<i\le z$, define

$$a_i=\begin{cases} j, & \text{if } \mathsf{U}^j_i=\mathsf{T}^j_i-1 \text{ for some } j,\\ a_{i+1}, & \text{otherwise.}\end{cases} \tag{3.16}$$

(In other words, U is obtained form T by moving an entry labeled k from row r to row z, then an entry labeled a_z from row z to row $z-1$ and so on, until an entry labeled a_{r+1} is moved from row $r+1$ into row r.) For $r \le i \le z-1$, define

$$g_{\mathsf{T}}^{\mathsf{U}}(i) = \begin{cases} [c_z - c_i - \gamma + j + \mathsf{U}_i^{a_{i+1}}], & \text{if } a_i = a_{i+1}, \\ [\mathsf{U}_i^{a_{i+1}^{\mathsf{U}}}], & \text{if } a_i < a_{i+1} \text{ or } i = r, \\ q^{c_z - c_i - \gamma + j}[\mathsf{U}_i^{a_{i+1}}], & \text{if } a_i > a_{i+1}. \end{cases}$$

Set $g_{\mathsf{T}}^{\mathsf{U}} = g_{\mathsf{T}}^{\mathsf{U}}(r) \dots g_{\mathsf{T}}^{\mathsf{U}}(z-1)$ and if $r \le x \le y \le z$, let $b_{\mathsf{U}}^{\mathsf{T}}(x,y) = \sum_{i=x}^{y-1} \mathsf{U}_i^{>a_{i+1}}$.

Lemma 3.17. *Suppose T is a basic tableau of type $\eta + 1^\gamma$ such that $\eta \ne \mu$. Let j be maximal such that $r = \mathrm{row}_{\mathsf{T}}(z+j) < z$. Then*

$$m_{\mathsf{T}} \prod_{i=r}^{z-1} \big(L_{n+j} - [c_i] \big) = q^{c_{r+1} + \dots + c_z - \gamma + j} \sum_{\mathsf{U} \prec_{z+j}^{\mathsf{w}} \mathsf{T}} q^{b_{\mathsf{U}}^{\mathsf{T}}(r,z)} g_{\mathsf{T}}^{\mathsf{U}} m_{\mathsf{U}}.$$

Proof. We use induction on $z-r$ combined with Corollary 3.13. If $r = z-1$ then the result follows from Corollary 3.13. Now suppose that $r < z-1$ and that Lemma 3.17 holds for $r < r' \le z$. Let $\mathcal{L}_{n+j} = \prod_{i=r}^{z-1} \big(L_{n+j} - [c_i] \big)$. Then by Corollary 3.13 and induction, it is clear that $m_{\mathsf{T}} \mathcal{L}_{n+j}$ is a linear combination of terms m_{U} where $U \prec_{z+j}^{\mathsf{w}} \mathsf{T}$.

For the remainder of this proof fix a tableau U such that $U \prec_{z+j}^{\mathsf{w}} \mathsf{T}$ and let $\mathbf{a} = \mathbf{a}_{\mathsf{T}}^{\mathsf{U}}$ be the sequence defined in (3.16) above. Set $a_{z+1} = \infty$ and let $v \ge r+1$ be minimal such that $a_v < a_{v+1}$. Define integers $r = r_0 < r_1 < r_2 < \dots < r_s = v$ to be the points at which $a_{r_\sigma} > a_{r_\sigma + 1}$, for $1 \le \sigma < s$. Then

$$a_{r_0+1} = \dots = a_{r_1} > a_{r_1+1} = \dots = a_{r_2} > \dots > a_{r_{s-1}+1} = \dots = a_{r_s},$$

and $a_{r_s} < a_{r_s+1}$. Finally, let $R = R_{\mathsf{T}}^{\mathsf{U}} = \{\, r_\sigma \mid 1 \le \sigma \le s \,\}$.

Suppose that $r+1 \le x \le v$. Then $r_{\epsilon-1} < x \le r_\epsilon$ for some $\epsilon = \epsilon(x)$, where $1 \le \epsilon \le s$. Define integers $r'_0, r'_1, \dots, r'_\epsilon$ and $d_1, \dots, d_\epsilon$ by setting $d_\sigma = a_{r+\sigma}$, for $1 \le \sigma \le \epsilon$, and $r'_\sigma = r_\sigma$, for $0 \le \sigma < \epsilon$, and put $r'_\epsilon = x$. Now define $\mathsf{V}(x)$ to be the tableau obtained from U by repeatedly exchanging $n+j$ in row r'_σ with $d_{\sigma+1}$ in row $r'_{\sigma+1}$. Then the set of tableaux $\{\, \mathsf{V} \mid \mathsf{U} \prec_{n+l}^{\mathsf{w}} \mathsf{V} \prec_{n+l} \mathsf{T} \,\}$ is precisely the set $\{\, \mathsf{V}(x) \mid r+1 \le x \le v \,\}$.

For this paragraph fix x with $r+1 \le x \le v$. For convenience we set $C_x = c_z - c_x - \gamma + j$ and $\mathsf{U}_x = \mathsf{U}_x^{a_{x+1}}$. Recall that $c_x^\eta = \eta_x - x$, that is, $c_x^\eta = c_x$ for $r+1 \le x < z$ and $c_z^\eta = c_z - \gamma + j$. Then, by Corollary 3.13, the coefficient of $m_{\mathsf{V}(x)}$ in $m_{\mathsf{T}}(L_{n+j} - [c_r])$ is

$$q^{b_{\mathsf{T}}^{\mathsf{V}(x)}} (q-1)^{\epsilon-1} f_{\mathsf{T}}^{\mathsf{V}(x)} = q^{c_x^\eta + b_r^x(\mathsf{U},\mathsf{T})} (q-1)^{\epsilon-1} \prod_{\substack{y=r+1 \\ y \notin R}}^{x-1} q^{\mathsf{U}_y} \cdot \prod_{\sigma=1}^{\epsilon-1} [\mathsf{U}_{r_\sigma}].$$

If $x \ne z$ then, by induction, the coefficient of m_{U} in $m_{\mathsf{V}(x)} \prod_{i=x}^{z-1}(L_{n+j} - [c_i])$ is

$$q^{c_{x+1} + \dots + c_z - \gamma + j + b_{\mathsf{U}}^{\mathsf{T}}(x,z)} [\mathsf{U}_x] \prod_{\tau=x+1}^{z-1} g_{\mathsf{T}}^{\mathsf{U}}(\tau).$$

Finally, by Lemma 3.14,

$$m_{\mathsf{U}} \prod_{i=r+1}^{x-1} (L_{n+j} - [c_i]) = q^{c_{r+1} + \dots + c_{x-1}} \prod_{y=r+1}^{x-1} [C_y] m_{\mathsf{U}}.$$

As already noted, $\{\, \mathsf{V} \mid \mathsf{U} \prec_{n+l}^{\mathsf{w}} \mathsf{V} \prec_{n+l} \mathsf{T} \,\} = \{\, \mathsf{V}(x) \mid 1 \le x \le v \,\}$. Assume now that $v \ne z$; the case $v = z$ is similar but contains some technical differences which we leave to the reader. Collecting the terms above, the coefficient of $q^{c_{r+1} + \dots + c_z - \gamma + j + b_{\mathsf{U}}^{\mathsf{T}}(r,z)} m_{\mathsf{U}}$

in $m_{\mathsf{T}}\mathcal{L}_{n+j}$ is

$$\begin{aligned}\sum_{x=r+1}^{v}(q-1)^{\epsilon(x)-1}[\mathsf{U}_x]\prod_{\substack{y=r+1\\ y\notin R}}^{x-1}q^{\mathsf{U}_y}\cdot\prod_{\sigma=1}^{\epsilon(x)-1}[\mathsf{U}_{r_\sigma}]\cdot\prod_{\tau=x+1}^{z-1}g^{\mathsf{U}}_{\mathsf{T}}(\tau)\cdot\prod_{y=r+1}^{x-1}[C_y]\\
=\prod_{y=v+1}^{z-1}g^{\mathsf{U}}_{\mathsf{T}}(y)\cdot\Bigg\{\sum_{x=r+1}^{v}[\mathsf{U}_x]\prod_{\substack{y=r+1\\ y\notin R}}^{x-1}q^{\mathsf{U}_y}[C_y]\cdot\prod_{\sigma=1}^{\epsilon(x)-1}(q^{C_{r_\sigma}}-1)[\mathsf{U}_{r_\sigma}]\cdot\prod_{\tau=x+1}^{v}g^{\mathsf{U}}_{\mathsf{T}}(\tau)\Bigg\},\end{aligned}$$

where the last equation follows by rearranging the terms using the identity $(q-1)[C]=q^C-1$, for any $C\in\mathbb{Z}$. For $1\le x\le v$ set

$$h(x)=[\mathsf{U}_x]\prod_{\substack{y=r+1\\ y\notin R}}^{x-1}q^{\mathsf{U}_y}[C_y]\cdot\prod_{\sigma=1}^{\epsilon(x)-1}(q^{C_{r_\sigma}}-1)[\mathsf{U}_{r_\sigma}]\cdot\prod_{y=x+1}^{v}g^{\mathsf{U}}_{\mathsf{T}}(y).$$

To complete the proof of the lemma we need to show that $\sum_{x=r+1}^{v}h(x)=\prod_{x=r}^{v}g^{\mathsf{U}}_{\mathsf{T}}(x)$. Hence, it is enough to establish the following claim and then set $\epsilon=1$:

Claim. *Suppose that* $1\le\epsilon\le s$. *Then*

$$\sum_{x=r_{\epsilon-1}+1}^{v}h(x)=\prod_{\substack{y=r+1\\ y\notin R}}^{r_{\epsilon-1}}q^{\mathsf{U}_y}[C_y]\cdot\prod_{\sigma=1}^{\epsilon-1}(q^{C_{r_\sigma}}-1)[\mathsf{U}_{r_\sigma}]\cdot\prod_{\tau=r_{\epsilon-1}+1}^{v}g^{\mathsf{U}}_{\mathsf{T}}(\tau).$$

We prove the claim by downwards induction on ϵ. If $\epsilon=s$ then $\epsilon(x)=s$, for $x=r_{s-1}+1,\dots,r_s=v$, so

$$\sum_{x=r_{s-1}+1}^{v}h(x)=\sum_{x=r_{s-1}+1}^{v}[\mathsf{U}_x]\prod_{\substack{y=r+1\\ y\notin R}}^{x-1}q^{\mathsf{U}_y}[C_y]\cdot\prod_{\sigma=1}^{s}(q^{C_{r_\sigma}}-1)[\mathsf{U}_{r_\sigma}]\cdot\prod_{\tau=x+1}^{v}g^{\mathsf{U}}_{\mathsf{T}}(\tau).$$

Consulting the definitions reveals that for $r+1\le y\le v$ we have

$$g^{\mathsf{U}}_{\mathsf{T}}(y)=\begin{cases}[U_y], & \text{if } y=v,\\ q^{C_y}[\mathsf{U}_y], & \text{if } v\ne y\in R,\\ [C_y+\mathsf{U}_y], & \text{if } y\notin R.\end{cases}$$

Therefore,

$$\begin{aligned}\sum_{x=r_{s-1}+1}^{v}h(x)&=[U_v]\cdot\prod_{\substack{y=r+1\\ y\notin R}}^{r_{\epsilon-1}}q^{\mathsf{U}_y}[C_y]\cdot\prod_{\sigma=1}^{s-1}(q^{C_{r_\sigma}}-1)\Bigg\{\prod_{y=r_{s-1}+1}^{v-1}q^{\mathsf{U}_y}[C_y]\\
&\qquad+\sum_{x=r_{s-1}+1}^{v-1}\prod_{y=r_{s-1}+1}^{x-1}q^{\mathsf{U}_y}[C_y]\cdot[\mathsf{U}_x]\cdot\prod_{y=x+1}^{v-1}[C_y+\mathsf{U}_y]\Bigg\}\\
&=[U_v]\cdot\prod_{\substack{y=r+1\\ y\notin R}}^{r_{\epsilon-1}}q^{\mathsf{U}_y}[C_y]\cdot\prod_{\sigma=1}^{s}(q^{C_{r_\sigma}}-1)\cdot\prod_{i=r_{s-1}+1}^{v-1}[C_i+\mathsf{U}_i]\end{aligned}$$

by Lemma 3.15. This proves the claim when $\epsilon=s$. The proof of the claim when $\epsilon<s$ follows easily by induction using a similar argument, so we leave the details to the reader. □

Corollary 3.18. *Suppose that* T *is a basic tableau and that* $j \in [1, \gamma]$ *is an integer such that either* $j = \gamma$ *or* $\operatorname{row}_{\mathsf{T}}(z+j+1) = z$. *Let* $r = \operatorname{row}_{\mathsf{T}}(n+j)$ *and fix* y *with* $1 \le y \le r$. *If* $r = z$ *then*

$$m_{\mathsf{T}} \prod_{i=y}^{z-1} (L_{n+j} - [c_i]) = q^{c_1 + \cdots + c_{z-1}} \prod_{i=y}^{z-1} [c_z - c_i - \gamma + j]\, m_{\mathsf{T}}.$$

If $r < z$ *then*

$$m_{\mathsf{T}} \prod_{i=y}^{z-1} (L_{n+j} - [c_i]) = q^{c_1 + \cdots + c_z - c_r + j - \gamma} \prod_{i=y}^{r-1} [c_z - c_i - \gamma + j] \sum_{\mathsf{U} \prec^{\mathrm{w}}_{n+j} \mathsf{T}} q^{b^{\mathsf{T}}_{\mathsf{U}}(r,z)} g^{\mathsf{U}}_{\mathsf{T}}\, m_{\mathsf{U}}.$$

Proof. This is an immediate consequence of Proposition 3.17 and Lemma 3.14. □

The next result will complete the proof of Theorem 2.7. Although we could prove this result for a slightly more general class of partitions, we assume that $\nu_i - \nu_{i+1} \ge \gamma$, for $1 \le i < z$, because this assumption significantly simplifies the notation that we need.

Suppose $\mathsf{t} = \mathsf{t}^{\nu}_{\eta}$ is an almost initial tableau. Choose k with $1 \le k \le \gamma$ and let $\eta^{(k)}$ be the partition of n given by

$$\eta^{(k)}_i = \begin{cases} \eta_i + \mathsf{t}_i^{>n+\gamma-k}, & 1 \le i < z, \\ \nu_i - k, & i = z. \end{cases}$$

Write $\mathsf{U} \xleftarrow{k} \mathsf{t}$ if $\mathsf{U} \in \mathcal{T}_0(\nu, \eta + 1^{\gamma})$ and $\operatorname{Shape}(\mathsf{U}_{\downarrow z}) = \eta^{(k)}$ and the numbers $z+1, z+2, \dots, z+\gamma$ in U are in row order.

Proposition 3.19. *Assume that* $\nu_i - \nu_{i+1} \ge \gamma$, *for* $1 \le i < z$, *and that* $\mathsf{t} = \mathsf{t}^{\nu}_{\eta}$ *is an almost initial tableau. Suppose that* $1 \le k \le \gamma$ *and that* $1 \le y \le \operatorname{row}_{\mathsf{t}}(n+\gamma-k+1)$. *Then*

$$m_{\mathsf{t}} \prod_{i=y}^{z} \prod_{j=1}^{k} (L_{n+\gamma-j+1} - [c_i]) = q^{c(k)} \sum_{\mathsf{U} \xleftarrow{k} \mathsf{t}} \left(\prod_{i=y}^{z-1} [\mathsf{U}^{(i,z]}_i]! \prod_{j=0}^{k-\mathsf{U}^{(i,z]}_i - 1} [c_z - c_i - j] \right) m_{\mathsf{U}}$$

where

$$c(k) = \sum_{i=y}^{z} k c_i + \mathsf{t}_i^{>n+\gamma-k} \Big(\mathsf{t}_i^{(n,n+\gamma-k]} - \mathsf{t}_{>i}^{>n+\gamma-k} - c_i \Big).$$

Proof. For the duration of the proof we set $\mathcal{L}'_{k'} = \prod_{i=y}^{z} \prod_{j=1}^{k'} (L_{n+\gamma-j+1} - [c_i])$, for $1 \le k' \le k$. Then we have to compute $m_{\mathsf{t}} \mathcal{L}'_k$. First note that if T is the basic tableau obtained by replacing each entry x with $1 \le x \le n$ in t by its row index in t and each entry $n+1 \le x \le n+\gamma$ with $x - n + z$ then $m_{\mathsf{t}} = m_{\mathsf{T}}$ by (3.4). If $k = 1$ or $\operatorname{row}_{\mathsf{t}}(n+\gamma-k+1) = z$ then the result follows from Corollary 3.18. So suppose that $1 < k \le \gamma$ and that $\operatorname{row}_{\mathsf{t}}(n+\gamma-k+1) = r < z$. By induction on k we can assume that the Proposition holds for $m_{\mathsf{t}} \mathcal{L}'_{k'}$ whenever $1 \le k' < k$.

Repeated applications of Corollary 3.18 shows that $m_{\mathsf{t}} \mathcal{L}'_k = m_{\mathsf{T}} \mathcal{L}'_k$ is a linear combination of terms m_{U}, where $\mathsf{U} \xleftarrow{k} \mathsf{t}$. That each tableau U is semistandard follows because $\nu_i - \nu_{i+1} \ge \gamma$ for all i. We now fix U with $\mathsf{U} \xleftarrow{k} \mathsf{t}$ and compute the coefficient of m_{U} in $m_{\mathsf{T}} \mathcal{L}'_k$.

Suppose that V is a basic tableau such that $\mathsf{U} \prec^{\mathrm{w}}_{n+\gamma-k+1} \mathsf{V} \xleftarrow{k-1} \mathsf{t}$. By Corollary 3.18, the coefficient of m_{U} in $m_{\mathsf{V}} \prod_{i=y}^{z-1} (L_{n+\gamma-k+1} - [c_i])$ is

$$q^{c_1 + \cdots + c_{r-1} + c_{r+1} + \dots + c_z + b^z_r(\mathsf{U},\mathsf{V}) - k + 1} g^{\mathsf{U}}_{\mathsf{V}} \prod_{i=y}^{r-1} [c_z - c_i - k + 1].$$

By induction, the coefficient of m_{V} in $m_{\mathfrak{t}}\mathcal{L}'_{k-1}$ is

$$q^{c(k-1)}\prod_{i=y}^{z-1}\Big([\mathsf{V}_i^{(i,z]}]^! \prod_{j=0}^{k-\mathsf{V}_i^{(i,z]}-1}[c_z-c_i-j]\Big).$$

Now observe that

$$c(k)=c(k-1)+c_1+\ldots+c_z-c_r+\mathfrak{t}^{(n,n+\gamma-k]}-k+1.$$

Therefore, the coefficient of $q^{c(k)}m_{\mathsf{U}}$ in $m_{\mathfrak{t}}\mathcal{L}'_k$ is

$$\sum_{\substack{\mathsf{V}\in\mathcal{T}_0(\nu,\eta+1^\gamma)\\ \mathsf{U}\prec^{\mathsf{w}}_{n+\gamma-k+1}\mathsf{V}\xleftarrow{k-1}\mathfrak{t}}} q^{\mathfrak{t}_r^{(n,n+\gamma-k]}+b_r^z(\mathsf{U},\mathsf{V})}g_{\mathsf{V}}^{\mathsf{U}}\prod_{i=y}^{r-1}[c_z-c_i-k+1]\cdot\prod_{i=y}^{z-1}\Big([\mathsf{V}_i^{(i,z]}]^!\prod_{j=0}^{k-\mathsf{V}_i^{(i,z]}-1}[c_z-c_i-j]\Big).$$

Consulting the definitions, if $\mathsf{V}\in\mathcal{T}_0(\nu,\eta+1^\gamma)$ and $\mathsf{U}\prec^{\mathsf{w}}_{n+\gamma-k+1}\mathsf{V}\xleftarrow{k-1}\mathfrak{t}$ then

$$\mathsf{V}_i^{(i,z]}=\begin{cases}\mathsf{U}_i^{(i,z]}, & 1\le i\le r-1,\ \text{or } r+1\le i\le z \text{ and } r+b_i\ne i,\\ \mathsf{U}_i^{(i,z]}-1, & i=r,\ \text{or } r+1\le i\le z \text{ and } b_i=i,\end{cases}$$

whenever $1\le i\le z$. This allows us to rewrite the last equation in terms of U. Before we do this, however, we change the indexing set for the sum to something that is more manageable.

Suppose that $\mathsf{U}\prec^{\mathsf{w}}_{n+\gamma-k+1}\mathsf{V}$. Then V is completely determined by a sequence $\mathbf{a}_{\mathsf{V}}^{\mathsf{U}}=(a_{r+1},\ldots,a_z)$ as in (3.16). Let $\mathcal{A}=\{\,\mathbf{a}=(a_{r+1},\ldots,a_z)\mid i\le a_i\le z \text{ for } r\le i\le z\,\}$. Then $\mathbf{a}_{\mathsf{V}}^{\mathsf{U}}\in\mathcal{A}$ for each tableau V in the sum above. Conversely, if $\mathbf{a}\in\mathcal{A}$ and $\mathbf{a}$ does not correspond to one of the tableau above then there exists an i, with $r\le i\le z-1$, such that $a_i\ne a_{i+1}$ and $\mathsf{U}_i^{a_{i+1}}=0$. Therefore, $h_{\mathsf{U}}^{\mathbf{a}}(i)=0$, where we define

$$h_{\mathsf{U}}^{\mathbf{a}}(i)=\begin{cases}[C_i+\mathsf{U}_i^{a_{i+1}}][\mathsf{U}_i^{(i,z]}], & \text{if } i\ne a_i=a_{i+1},\\ [C_i+\mathsf{U}_i^{(i,z]}][\mathsf{U}_i^{a_{i+1}}], & \text{if } i=a_i<a_{i+1},\ \text{or if } i=r,\\ [\mathsf{U}_i^{a_{i+1}}][\mathsf{U}_i^{(i,z]}], & \text{if } i\ne a_i<a_{i+1},\\ q^{C_i}[\mathsf{U}_i^{a_{i+1}}][\mathsf{U}_i^{(i,z]}], & \text{if } i\ne a_i>a_{i+1}.\end{cases}$$

where $C_i=c_z-c_i-k+1$, for $r\le i<z$. Recall that $b_r^z(\mathsf{U},\mathsf{V})=\sum_{i=r}^{z-1}\mathsf{U}_i^{>a_{i+1}}=\sum_{i=r}^{z-1}\mathsf{U}_i^{(a_{i+1},z]}+\mathfrak{t}_r^{(n,n+\gamma-k]}$. Therefore, by comparing the definitions of $g_{\mathsf{U}}^{\mathsf{V}}(i)$ and $h_{\mathsf{U}}^{\mathbf{a}}(i)$, and observing that $\mathsf{V}_i^{(i,l)}\le\mathsf{U}_i^{(i,l)}-1$, the coefficient of $q^{c(k)}m_{\mathsf{U}}$ in $m_{\mathfrak{t}}\mathcal{L}'_k$ given above becomes

$$\prod_{i=y}^{r-1}[C_i]\cdot\prod_{i=y}^{z-1}\Big([\mathsf{U}_i^{(i,z]}-1]^!\prod_{j=0}^{k-\mathsf{U}_i^{(i,z]}-2}[c_z-c_i-j]\Big).\sum_{\mathbf{a}\in\mathcal{A}}\prod_{i=r}^{z-1}q^{\mathsf{U}_i^{(a_{i+1},z]}}h_{\mathsf{U}}^{\mathbf{a}}(i)$$

where we adopt the convention that $[-1]^!=1$. By definition, $\mathsf{U}_i^{(i,z]}=0$, for $1\le i<r$, and $C_i+\mathsf{U}_i^{(i,z]}=c_z-c_i-(k-\mathsf{U}_i^{(i,z]}-1)$, for $1\le i<z$. Therefore, to complete the proof we need to show that

$$\sum_{\mathbf{a}\in\mathcal{A}}\prod_{i=r}^{z-1}q^{\mathsf{U}_i^{(a_{i+1},z]}}h_{\mathsf{U}}^{\mathbf{a}}(i)=\prod_{i=r}^{z-1}[\mathsf{U}_i^{(i,z]}][C_i+\mathsf{U}_i^{(i,z]}].$$

This will follow once we have established the following claim by setting $x=r$ and, for definiteness, $a=r$.

Claim. *Let* $\mathcal{A}_{a,x} = \{\, (a, a_{x+1}, \dots, a_z) \mid i \le a_i \le z$ *for* $x+1 \le i \le z \,\}$ *where* $r \le x \le z-1$ *and* $x \le a \le z$*. Then*

$$\sum_{\mathbf{a}\in\mathcal{A}_{a,x}} \prod_{i=x}^{z-1} q^{\mathsf{U}_i^{(a_{i+1},z]}} h_{\mathsf{U}}^{\mathbf{a}}(i) = \prod_{i=x}^{z-1} [\mathsf{U}_i^{(i,z]}][C_i + \mathsf{U}_i^{(i,z]}].$$

To prove the claim, we use downwards induction on x. If $x = z-1$ then $b = z-1$ or $b = z$. If $a = z-1$ or $x = r$ then

$$\sum_{\mathbf{a}\in\mathcal{A}_{a,x}} \prod_{i=x}^{z-1} q^{\mathsf{U}_i^{(a_{i+1},z]}} h_{\mathsf{U}}^{\mathbf{a}}(i) = [C_{z-1} + \mathsf{U}_{z-1}^{[z,z]}][\mathsf{U}_{z-1}^{z}],$$

and if $a = z$ and $x \ne r$ then

$$\sum_{\mathbf{a}\in\mathcal{A}_{a,x}} \prod_{i=x}^{z-1} q^{\mathsf{U}_i^{(a_{i+1},z]}} h_{\mathsf{U}}^{\mathbf{a}}(i) = [C_{z-1} + \mathsf{U}_{z-1}^{z}][\mathsf{U}_{z-1}^{[z,z]}].$$

Since $\mathsf{U}_{z-1}^{z} = \mathsf{U}_{z-1}^{[z,z]}$, the claim holds for $x = z-1$. So suppose $r+1 \le x < z-1$ and the claim holds for $x+1$.

$$\begin{aligned}\sum_{\mathbf{a}\in\mathcal{A}_{a,x}} \prod_{i=x}^{z-1} q^{\mathsf{U}_i^{(a_{i+1},z]}} h_{\mathsf{U}}^{\mathbf{a}}(i) &= \sum_{a_x=x+1}^{z} q^{\mathsf{U}_x^{(a_{x+1},z]}} h_{\mathsf{U}}^{\mathbf{a}}(x) \sum_{\mathbf{a}\in\mathcal{A}_{a,x+1}} \prod_{i=x+1}^{z-1} q^{\mathsf{U}_i^{(a_{i+1},z]}} h_{\mathsf{U}}^{\mathbf{a}}(i)\\ &= \prod_{i=x+1}^{z-1} [\mathsf{U}_i^{(i,z]}][C_i + \mathsf{U}_i^{(i,z]}] \sum_{a_{x+1}=x+1}^{z} q^{\mathsf{U}_x^{(a_{x+1},z]}} h_{\mathsf{U}}^{\mathbf{a}}(x)\end{aligned}$$

by induction. If $a = x$ or $x = r$ then

$$\begin{aligned}\sum_{a_{x+1}=x+1}^{z} q^{\mathsf{U}_x^{(a_{x+1},z]}} h_{\mathsf{U}}^{\mathbf{a}}(x) &= \sum_{a_{x+1}=x+1}^{z} q^{\mathsf{U}_x^{(a_{x+1},z]}} [\mathsf{U}_x^{x_{a+1}}]\\ &= [\mathsf{U}_x^{(x,z]}].\end{aligned}$$

If $a \ne x$ and $x \ne r$ then $\sum_{a_{x+1}=x+1}^{z} q^{\mathsf{U}_x^{(a_{x+1},z]}} h_{\mathsf{U}}^{\mathbf{a}}(x)$ is equal to

$$\begin{aligned}&[\mathsf{U}_x^{(x,z]}]\left(\sum_{i=x+1}^{a-1} q^{C_x} q^{\mathsf{U}_x^{(i,z]}} [\mathsf{U}_x^{i}] + q^{\mathsf{U}_x^{(a,z]}} [C_x + \mathsf{U}_x^{a}] + \sum_{i=a+1}^{z} q^{\mathsf{U}_x^{(i,z]}} [\mathsf{U}_x^{i}] \right)\\ &\qquad = [\mathsf{U}_x^{(x,z]}][C_x + \mathsf{U}_x^{(x,z]}]\end{aligned}$$

This completes the proof of both the claim and the Proposition. □

As Proposition 2.5 is a special case of Proposition 3.19, this completes the proof of Theorem 2.7 and, in fact, all of our main results when F is a field of characteristic zero.

3.3. **Gaussian integer division.** In this section we prove Lemma 3.24 which were used in Section 2 to define the polynomials $\beta_{\lambda\mu}(q)$ in (2.9). Therefore, the results in this subsection complete the proof of our main results when F is a field of positive characteristic. Accordingly, we assume that F is a field of characteristic $p > 0$, that $e > 1$ and that ζ is a primitive e^{th} root of unity in F.

Let $K = F(q)$, where q is an indeterminate over F. For $l \in \mathbb{Z}$, set $[l]_q = \frac{q^l - 1}{q-1} \in K$. Set $[0]_q^! = 1 \in K$ and for $l \ge 1$ set $[l]_q^! = [l-1]_q^! [l]_q$. For $l \in \mathbb{Z} \setminus \{0\}$, define $\nu_p(l)$ to be the largest integer $v \ge 0$ such that p^v divides l (in $\mathbb{Z}$) and set

$$\nu_{e,p}(l) = \begin{cases} 0, & \text{if } e \nmid l,\\ 1 + \nu_p(\frac{l}{e}), & \text{otherwise.}\end{cases}$$

Lemma 3.20. *Suppose that $r \geq 1$ and that $(a_1, a_2, \dots, a_r)$ and $(b_1, b_2, \dots, b_r)$ are two r-tuples of non-zero integers such that $\nu_{e,p}(a_j) \geq \nu_{e,p}(b_j)$, for $1 \leq j \leq r$. Then there exist polynomials $f(q), g(q) \in F[q, q^{-1}]$ such that $g(\zeta) \neq 0$ and*

$$\frac{\prod_{j=1}^r [a_j]_q}{\prod_{j=1}^r [b_j]_q} = \frac{f(q)}{g(q)}.$$

Proof. It is sufficient to show that if $a, b \in \mathbb{Z} \setminus \{0\}$ and $\nu_{e,p}(a) \geq \nu_{e,p}(b)$ then $[a]_q/[b]_q$ can be written in this form. Since $[l]_q = -q^l[-l]_q$, we may assume that $a, b > 0$. If $\nu_{e,p}(b) = 0$ then $[a]_q/[b]_q$ itself is of the correct form. So take $a = xep^k, b = yep^l$ where $p \nmid x, y$ and $k \geq l$. Then

$$\frac{[a]_q}{[b]_q} = \frac{1 + q + \ldots + q^{a-1}}{1 + q + \ldots + q^{b-1}} = \frac{1 + q^{ep^l} + \ldots + q^{(xp^{k-l}-1)ep^l}}{1 + q^{ep^l} + \ldots + q^{(y-1)ep^l}}.$$

Since ζ is an e^{th} root of unity and $p \nmid y$, the value of the denominator of the right hand term at ζ is non-zero. □

Lemma 3.21. *Suppose that $K, \gamma, m > 0$. For any integer l define l' by writing $l = l^* m + l'$ where $0 \leq l' < m$. Let $C = -K$. For $0 \leq X \leq \gamma$, let $\mathcal{M}_X$ be the multiset $\{1, 2, \dots, X, K, K+1, \dots, K+\gamma-X-1\}$ and let $N(X)$ be the number of elements of $\mathcal{M}_X$ which are divisible by m. Then*

$$N(X) = \begin{cases} \max\left\{0, \left\lceil \frac{\gamma - C'}{m} \right\rceil\right\}, & X' < (\gamma+K)', \\ \max\left\{0, \left\lfloor \frac{\gamma - C'}{m} \right\rfloor\right\}, & X' \geq (\gamma+K)'. \end{cases}$$

Proof. By definition, $N(X)$ is equal to the number of elements of $\{K, K+1, \dots, K+\gamma-X'-1\}$ which are divisible by m. It is then straightforward to check that this

$$N(X) = \max\left\{0, \left\lceil \frac{\gamma - C' - X'}{m} \right\rceil\right\}.$$

Noting that $(\gamma - C')' = (\gamma + K)'$, the result follows. □

Lemma 3.22. *Suppose $K > 0$ and $\gamma \geq e$. For $0 \leq X \leq \gamma$, let $\mathcal{M}_X$ be the multiset*

$$\mathcal{M}_X = \{1, 2, \dots, X, K, K+1, \dots, K+\gamma-X-1\}.$$

For $i \geq 0$, set $N(X)_i = \#\{\, x \in \mathcal{M}_X \mid \nu_{e,p}(x) \geq i \,\}$. Let s be maximal such that $\gamma \geq ep^s$ and A minimal such that $Aep^s \geq K$ and set $\beta = \gamma - Aep^s + K$, so that

$$\mathcal{M}_\beta = \{1, 2, \dots, \gamma - Aep^s + K, K, K+1, \dots, Aep^s - 1\}.$$

Then $0 \leq \beta \leq \gamma$ and if $0 \leq X \leq \gamma$ then $N(\beta)_i \leq N(X)_i$, for all $i \geq 0$.

Proof. That $0 \leq \beta \leq \gamma$ is clear from the definitions. To prove the second claim $i \geq 0$. For any integer $l \geq 0$ define l' by $l = l^* ep^i + l'$ where $0 \leq l' < ep^i$. By Lemma 3.21, to show that $N(\beta)_i \leq N(X)_i$ whenever $0 \leq X \leq \gamma$ it is sufficient to prove that $\beta' \geq (\gamma+K)'$. In fact, our choice of β gives $\beta' = (\gamma+K)'$. □

Corollary 3.23. *Suppose that $\gamma > 0$ and $C < 0$. For $0 \leq X \leq \gamma$, let $\mathcal{M}_X$ denote the multiset $\mathcal{M}_X = \{1, 2, \dots, X, C, C-1, \dots, C-\gamma+X+1\}$. For $i \geq 0$ let*

$$N(X)_i = \#\{\, x \in \mathcal{M}_X \mid \nu_{e,p}(x) \geq i \,\}.$$

Then there exists an integer β with $0 \leq \beta \leq \gamma$ such that $N(\beta)_i \leq N(X)_i$ whenever $0 \leq X \leq \gamma$ and $i \geq 0$.

Proof. If $\gamma < e$ then set $\beta = \gamma$. Otherwise set $K = -C$. Then for all i, $N(X)_i$ is the number of elements $x \in \{1, 2, \dots, X, K, K+1, \dots, K+\gamma-X-1\}$ such that $\nu_{e,p}(x) \geq i$. Hence, the result follows from Lemma 3.22. □

Lemma 3.24. *Suppose that $\gamma > 0$ and that $C < 0$. Write $\gamma = \gamma^* e + \gamma'$ where $0 \leq \gamma' < e$. Then there exists an integer β, with $0 \leq \beta \leq \gamma$, and polynomials $f_X(q), g_X(q) \in F[q, q^{-1}]$ such that $g_X(\zeta) \neq 0$ and*

$$\frac{[X]_q^! \prod_{j=0}^{\gamma-X-1} [C-j]_q}{[\beta]_q^! \prod_{j=0}^{\gamma-\beta-1} [C-j]_q} = \frac{f_X(q)}{g_X(q)},$$

whenever $0 \leq X \leq \gamma$. Moreover, if $C \equiv 0 \mod ep^{\ell_p(\gamma^*)}$ then $\beta = \gamma$ and $f_X(\zeta) \neq 0$ if and only if $X = \gamma$.

Proof. Using the notation of Corollary 3.23, there exists an integer β with $0 \leq \beta \leq \gamma$ such that $N(\beta)_i \leq N(X)_i$ for all $i \geq 0$. Therefore it is possible to reorder the elements in the multisets $\mathcal{M}_X = \{x_1, x_2, \dots, x_\gamma\}$ and $\mathcal{M}_\beta = \{b_1, b_2, \dots, b_\gamma\}$ in such a way that $\nu_{e,p}(x_j) \geq \nu_{e,p}(b_j)$, for $1 \leq j \leq \gamma$. Hence, by Lemma 3.20, there exists an integer β with the required properties.

Now suppose that $C \equiv 0 \mod ep^{\ell_p(\gamma^*)}$. Note that $ep^{\ell(\gamma^*)} > \gamma$. By Lemma 3.22, we may take $\beta = \gamma$. Now, suppose $X \neq \beta$. Reorder $\mathcal{M}_X$ and $\mathcal{M}_\beta$ as above so that $\nu_{e,p}(x_j) \geq \nu_{e,p}(b_j)$ for $1 \leq j \leq \gamma$. Assume that $x_1 = C$. By Lemma 3.20

$$\frac{\prod_{j=1}^{\gamma} [x_j]}{\prod_{j=1}^{\gamma} [b_j]} = \frac{[C]_q f'_X(q)}{[b_1]_q g'_X(q)}$$

for some $f'_X(q), g'_X(q) \in F[q, q^{-1}]$ with $g'_X(\zeta) \neq 0$. Since $1 \leq b_1 \leq \gamma$, we have $\nu_{e,p}(b_1) < \nu_{e,p}(C)$. Consider $[C]_q/[b_1]_q$. If $e \nmid b_1$ then the evaluation of $[C]_q$ at ζ is zero. Otherwise, write $-C = xep^k, b_1 = yep^l$ where $p \nmid x, y$ so that $k > l$. Then

$$\frac{[C]_q}{[b_1]_q} = \frac{-q^{-C}(1 + q^{ep^l} + \dots + q^{(xp^{k-l}-1)ep^l})}{1 + q^{ep^l} + \dots + q^{(y-1)ep^l}}.$$

Since $p \mid xp^{k-l}$, the numerator of the last term evaluated at ζ is zero. □

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